Massachusetts Institute of Technology
Department of Physics
8.962 Spring 2006

Problem Set 1
Post date: Thursday, February 9th
Due date: Thursday, February 16th

1. (a) [5 pts] Show that the sum of any two orthogonal spacelike vectors is spacelike.
   (b) [5 pts] Show that a timelike vector and a null vector cannot be orthogonal.

2. In some reference frame, the vector fields $\vec{U}$ and $\vec{D}$ have the components

$$U^\alpha \doteq (1+t^2, t^2, \sqrt{2}t, 0)$$
$$D^\alpha \doteq (x, 5tx, \sqrt{2}t, 0)\ .$$

The scalar ρ has the value

$$\rho = x^2 + t^2 - y^2\ .$$

(The relationship "LHS $\doteq$ RHS" means "the object on the left-hand side is represented by the object on the right-hand side in the specified reference frame.")

(a) [3 pts] Show that $\vec{U}$ is suitable as a 4-velocity. Is $\vec{D}$?

(b) [3 pts] Find the spatial velocity $\mathbf{v}$ of a particle whose 4-velocity is $\vec{U}$, for arbitrary t. Describe the motion in the limits $t = 0$ and $t \to \infty$.

(c) [3 pts] Find $\partial_\beta U^\alpha$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$. (There's a clever way to do this; do it the brute force way instead.)

(d) [3 pts] Find $\partial_\alpha D^\alpha$.

(e) [3 pts] Find $\partial_\beta(U^\alpha D^\beta)$ for all α.

(f) [3 pts] Find $U_\alpha \partial_\beta(U^\alpha D^\beta)$. Why is the answer so similar to that for (d)?

(g) [3 pts] Calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$.

(h) [3 pts] Find $\nabla_{\vec{U}}\rho$ and $\nabla_{\vec{D}}\rho$.

3. Consider a timelike unit 4-vector $\vec{U}$ and the tensor

$$P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta \ .$$

Show that this tensor is a projection operator that projects an arbitrary vector $\vec{V}$ into one orthogonal to $\vec{U}$. In other words, show that the vector $\vec{V}_\perp$ whose components are

$$V_\perp^\alpha = P^\alpha{}_\beta V^\beta$$

is

(a) [5 pts] orthogonal to $\vec{U}$

(b) [5 pts] unaffected by further projections:

$$V_{\perp\perp}^\alpha \equiv P^\alpha{}_\beta V_\perp^\beta = V_\perp^\alpha \ .$$

(c) [5 pts] Show that $P_{\alpha\beta}$ is the metric for the space of vectors orthogonal to $\vec{U}$:

$$P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot \vec{W}_\perp \ .$$

(d) [5 pts] Show that for an arbitrary nonnull vector $\vec{q}$, the projection tensor is given by

$$P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma} \ .$$

Do we need a projection tensor for null vectors?

4. [15 pts] Let $\Lambda_B(\mathbf{v})$ be a Lorentz boost associated with 3-velocity $\mathbf{v}$. Consider

$$\Lambda \equiv \Lambda_B(\mathbf{v}_1) \cdot \Lambda_B(\mathbf{v}_2) \cdot \Lambda_B(-\mathbf{v}_1) \cdot \Lambda_B(-\mathbf{v}_2)$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Assume $v_1 \ll 1$, $v_2 \ll 1$.

Show that Λ is a rotation. What is the axis of rotation? What is the angle of rotation?

5. "Superluminal" motion

   The quasar 3C 273 emits relativistic blobs of plasma from near the massive black hole at its center. The blobs travel at speed v along a jet making an angle θ with respect to the line of sight of the observer. Projected onto the sky, the blobs appear to travel perpendicular to the line of sight with angular speed $v_{\rm app}/r$ where r is the distance to 3C 273 as and $v_{\rm app}$ is the apparent speed.

   (a) [7 pts] Show that

   $$v_{\rm app} = \frac{v \sin\theta}{1 - v\cos\theta}\ .$$

   (b) [5 pts] For a given value of v, what value of θ maximizes $v_{\rm app}$? What is the corresponding maximal value of $v_{\rm app}$? Can this be greater than the speed of light? If so, is special relativity violated?

   (c) [3 pts] For 3C 273, $v_{\rm app} \simeq 10c$. What is the largest possible value of θ (in degrees)?

6. GZK cutoff in the cosmic ray spectrum

   (a) [8 pts] Calculate the threshold energy of a nucleon N for it to undergo the reaction $\gamma + N \to N + \pi^0$, where γ represents a microwave background photon of energy kT with $T = 2.73$ K. Assume the collision is head-on and take the nucleon and pion masses to be 938 MeV and 135 MeV, respectively.

   (b) [5 pts] Explain why one might expect to observe very few cosmic rays of energy above $\sim 10^{11}$ GeV.

   (c) [3 pts] This expectation is called the Griesen-Zatsepin-Kuzmin (GZK) cutoff. Modern observations show no sharp cutoff; there may even be evidence for an *upturn* in cosmic ray flux at these energies. Can you suggest a mechanism by which the GZK cutoff can be avoided?