

# Optimization Methods in Management Science

MIT 15.053, Spring 2013

PROBLEM SET 2 (SECOND GROUP OF STUDENTS)

Students with first letter of surnames **I–Z**

DUE: FEBRUARY 21 , 2013

## Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.
3. Late assignments will *not* be accepted. No exceptions.
4. The non-Excel solution should be handed in at the beginning of class on the day the problem set is due. The Excel solutions, if required, should be posted on the website by the beginning of class on the day the problem set is due. Questions that require an Excel submission are marked with EXCEL SUBMISSION . For EXCEL SUBMISSION questions, only the Excel spreadsheet will be graded.

## Problem 1

(33 points total) Consider the following linear program:

$$\left. \begin{array}{ll} \max & x_1 + 2x_2 \\ \text{s.t.:} & \\ \text{Const 1} & x_1 - x_2 \geq -2 \\ \text{Const 2} & x_1 + x_2 \leq 4 \\ \text{Const 3} & x_1 \leq 2.5 \\ \text{Const 4} & x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array} \right\} \quad (\text{LP})$$

- (a) (5 points) Graph the feasible region of the LP. Is the feasible region unbounded?

**Solution.** The feasible region and objective function are sketched in Figure 1. No. The feasible region is bounded.

- (b) (4 points) Are any of the above constraints redundant? If so, indicate which one(s). (For large linear programs, eliminating redundant constraints can speed up the solution of the linear program.)

**Solution.** Yes, the constraint  $x_2 \leq 3$  is redundant since deleting this constraint does not change the feasible region.

- (c) (4 points) Solve the LP using the graphical method. Explain your approach.

**Solution.** The optimal solution is clearly the point with coordinates (1, 3).

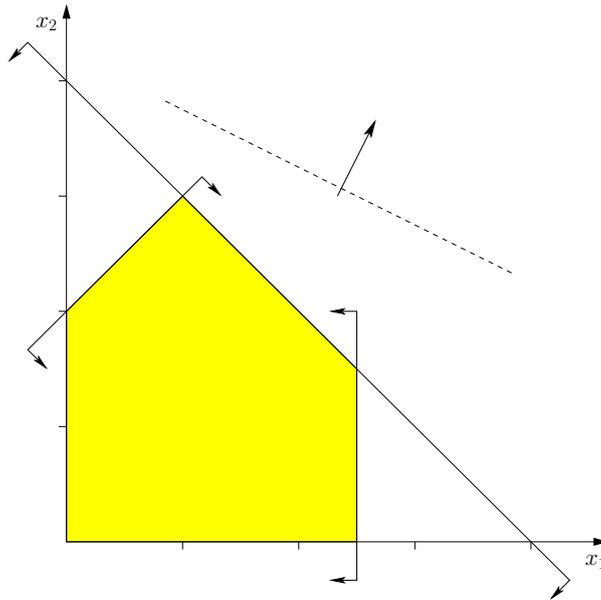


Figure 1: Feasible region for Part 1.A.

- (d) (4 points) Is there more than one optimal solution? If so give two different solutions. If not, explain using the graphical method why not?
- (e) (9 points, 3 points each) Suppose we add the constraint  $2x_1 + x_2 \geq \alpha$  to (LP). For which values of  $\alpha$ :
- is the constraint redundant?
  - the optimal solution found above is no longer optimal?
  - the problem becomes infeasible?

To answer these questions, you should use the graph of the feasible region drawn in Part (a).

**Solution.** The constraint  $2x_1 + x_2 \geq \alpha$  has slope -2, and the feasible halfspace is the one lying to the right of the line with equation  $2x_1 + x_2 = \alpha$ . Therefore it is easy to see that the constraint is redundant if  $2x_1 + x_2 = \alpha$  passes through the point  $(0,0)$  or if the intercept on the  $x_1$  axis is negative. This happens for  $\alpha \in [-\infty, 0]$  (for  $\alpha = 0$ , the line goes through the origin). A sketch of the situation is given in Figure 2. Note that the value  $\alpha = 0$  is acceptable. The solution  $(1,3)$  is no longer optimal if it is cut off from the feasible region by the constraint. The line  $2x_1 + x_2 = \alpha$  passes through  $(1,3)$  for  $\alpha = 5$ . Therefore the solution is no longer optimal for  $\alpha \in (5, \infty]$  (the value  $\alpha = 5$  is excluded). Finally, the problem is infeasible if all feasible points are cut off. The last point to be cut off is  $(2.5, 1.5)$ , for  $\alpha > 6.5$ . Hence the problem is infeasible for  $\alpha \in (6.5, \infty]$ .

- (f) (4 points) Replace the objective function  $x_1 + 2x_2$  with the objective function  $x_1 + \beta x_2$ , and compute the values of  $\beta$  for which the point  $(2.5, 1.5)$  is optimal.

**Solution.** Starting from the original objective function  $x_1 + 2x_2$ , which has slope  $-1/2$ , we easily see that we have to decrease the slope of the objective function in order to make

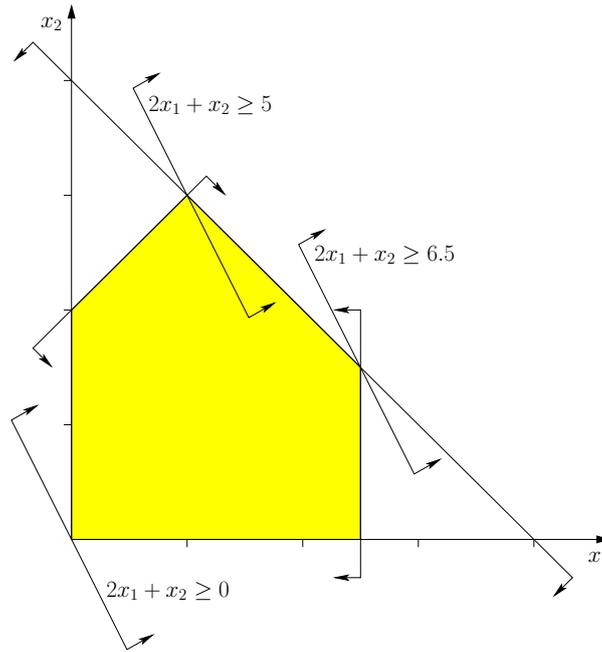


Figure 2: Initial feasible region plus constraint  $2x_1 + x_2 \geq \alpha$  for Part 1.B.

$(2.5, 1.5)$  optimal. Notice that the slope is equal to  $-1/\beta$ . When  $\beta = 1$ , the objective function is parallel to the constraint  $x_1 + x_2 \leq 4$  with slope  $-1$ , and the whole segment between  $(1, 3)$  and  $(2.5, 1.5)$  is optimal. In order to decrease the slope from  $-1$ , we have to decrease  $\beta$ . As  $\beta$  approaches 0, the objective function becomes a vertical line and the whole segment between  $(2.5, 0)$  and  $(2.5, 1.5)$  is optimal.

It follows that  $(2.5, 1.5)$  is optimal for  $\beta \in [0, 1]$ .

## Problem 2 <sup>1</sup>

(38 points total) A company makes three lines of tires. Its four-ply biased tires produce \$6 in profit per tire; its fiberglass belted line \$4 a tire; and its radials \$8 a tire. Each type of tire passes through three manufacturing stages as a part of the entire production process.

Each of the three process centers has the following hours of available production time per day:

	<i>Process</i>	<i>Hours</i>
1	Modeling	12
2	Curing	14
3	Assembly	16

The time required in each process to produce one hundred tires of each line is as follows:

Tire	Hours per 100 units		
	Modeling	Curing	Assembly
Four-ply	2	3	2
Fiberglass	2	2	1
Radial	4	2	2

<sup>1</sup>This problem is based on Problem 6 of Applied Mathematical Programming, Chapter 2.

- (a) (5 points) Write a linear program to determine the optimum product mix for each day's production, assuming all tires are sold.

**Solution.** We define 3 decision variables that stand for the number of tires of each type to be produced each day:  $x_{FP}$  for the Fours-ply,  $x_{FG}$  for Fiberglass, and  $x_R$  for radial. The problem can be formulated as follows:

$$\left. \begin{array}{ll} \max & 6x_{FP} + 4x_{FG} + 8x_V \\ \text{s.t.} & \\ \text{Modeling :} & 0.02x_{FP} + 0.02x_{FG} + 0.04x_R \leq 12 \\ \text{Curing :} & 0.03x_{FP} + 0.02x_{FG} + 0.02x_R \leq 14 \\ \text{Assembly :} & 0.02x_{FP} + 0.01x_{FG} + 0.02x_R \leq 16 \\ & x_{FP}, x_{FG}, x_R \geq 0. \end{array} \right\}$$

- (b) (10 points) EXCEL SUBMISSION Solve the problem using the simplex algorithm, employing the Excel spreadsheet given with this problem set. Make sure you understand the formulation and the meaning of the variables, and fill in the missing coefficients of the tableau. The spreadsheet that you submit should contain the sequence of tableaus that leads to the optimal tableau *only*. You will then use the same spreadsheet to help through the rest of Problem 2, but make sure you submit a spreadsheet containing the sequence of tableaus leading to the optimal tableau for the *original* data!
- (c) (8 points) Using the Excel spreadsheet to carry out the calculations, answer the following question:
- (1) What is the initial feasible solution? Give the value for all the decision variables and all the slack variables.
  - (2) What is the optimal solution? Give its objective function value and the value of all the decision variables.

**Solution.** The initial feasible solution is  $x_{FP} = x_{FG} = x_R = 0, s_1 = 12, s_2 = 14, s_3 = 16$ . The optimal solution is  $x_{FP} = 400, x_{FG} = 0, x_R = 100, s_1 = 0, s_2 = 0, s_3 = 6$ , with objective function value 3200.

- (d) (15 points, 5 points each) Using the Excel spreadsheet from the previous point to recompute the optimal solution of the LP, answer the following questions:
- (i) Suppose we increase the number of modeling hours per day from 12 to 13. How much does the profit increase? (Comment: This is the *shadow price* of the constraint on the modeling hours. Read the tutorial on "Sensitivity Analysis in 2 Dimensions" to learn more about shadow prices.)
  - (ii) What would be the increase in the profit if we increase the number of assembly hours per day from 16 to 17 (Assume that the number of modeling hours per day is 12. )
  - (iii) (Extra credit) Consider the answer to the previous two questions. What relationship do you observe between the shadow price of a constraint and the value of the corresponding slack variable in the optimal solution?

**Solution.**

- (i) The shadow price for the Modeling constraint is \$150 per hour of production. Therefore we could pay as much as \$150 in order to increase our production capabilities in terms of Modeling time by one hour. Any larger amount would not be profitable.
- (ii) The shadow price for the Assembly constraint is 0. It is not profitable to pay to increase the number of Assembly hours.
- (iii) Constraints such that the corresponding slack variable has zero value in the optimal solution (i.e. tight constraints) have nonzero shadow price. Constraints such that the corresponding slack variable has nonzero value in the optimal solution (i.e. non-tight constraints) have zero shadow price.

### Problem 3 (Second group of students) <sup>2</sup>

(34 points total) A corporation that produces gasoline and oil specialty additives purchases four grades of petroleum distillates, A, B, C, and D. The company then combines the four distillates to make three mixtures (Deluxe, Standard, and Economy) according to specifications of the maximum and/or minimum percentages of grades A, C, or D in each blend, given in Table 1.

Mixture	Max % allowed for Additive A	Min % allowed for Additive C	Max % allowed for Additive D	Selling price \$/gallon
Deluxe	60%	20%	10%	7.9
Standard	15%	60%	25%	6.9
Economy	–	50%	45%	5.0

Table 1: Specifications of the three mixtures.

Supplies of the three basic additives and their costs are given in Table 2.

Distillate	Max quantity available per day (gals)	Cost \$/gallon
A	4000	0.60
B	5000	0.52
C	3500	0.48
D	5500	0.35

Table 2: Supplies and costs of petroleum grades.

We write a linear program to determine the production policy that maximizes profits. To do that, we define twelve decision variables to indicate the gallons of each of the basic petroleum grades (additives) that contribute to the production of the three mixtures. We use the following labels:  $x_{AD}$  indicates the quantity of additive A that is used for production of mixture Deluxe,  $x_{AS}$  indicates the quantity of additive A that is used for production of mixture Standard,  $x_{AE}$  indicates the quantity of additive A that is used for production of mixture Economy. Similarly, we use  $x_{BD}, x_{BS}, x_{BE}$  to label the quantity of additive B used in each of the three mixtures,  $x_{CD}, x_{CS}, x_{CE}$  for additive C, and  $x_{DD}, x_{DS}, x_{DE}$  for additive D. The gallons of mixture Deluxe produced can be computed as  $x_{AD} + x_{BD} + x_{CD} + x_{DD}$ ; similarly for mixture Standard and Economy. The objective function is the total profit, computed as revenues minus costs.

<sup>2</sup>This problem is based on Problem 11 of Applied Mathematical Programming, Chapter 1.

The problem can be formulated as

$$\begin{array}{ll}
 \max & 7.3x_{AD} + 7.38x_{BD} + 7.42x_{CD} + 7.55x_{DD} \\
 & 6.3x_{AS} + 6.38x_{BS} + 6.42x_{CS} + 6.55x_{DS} \\
 & 4.4x_{AE} + 4.48x_{BE} + 4.52x_{CE} + 4.65x_{DE} \\
 \text{s.t.} & \\
 \text{Availability A :} & x_{AD} + x_{AS} + x_{AE} \leq 4000 \\
 \text{Availability B :} & x_{BD} + x_{BS} + x_{BE} \leq 5000 \\
 \text{Availability C :} & x_{CD} + x_{CS} + x_{CE} \leq 3500 \\
 \text{Availability D :} & x_{DD} + x_{DS} + x_{DE} \leq 5500 \\
 \text{Mixture D Add A :} & 0.4x_{AD} - 0.6x_{BD} - 0.6x_{CD} - 0.6x_{DD} \leq 0 \\
 \text{Mixture D Add C :} & -0.2x_{AD} - 0.2x_{BD} + 0.8x_{CD} - 0.2x_{DD} \geq 0 \\
 \text{Mixture D Add D :} & -0.1x_{AD} - 0.1x_{BD} - 0.1x_{CD} + 0.9x_{DD} \leq 0 \\
 \text{Mixture S Add A :} & 0.85x_{AS} - 0.15x_{BS} - 0.15x_{CS} - 0.15x_{DS} \leq 0 \\
 \text{Mixture S Add C :} & -0.6x_{AS} - 0.6x_{BS} + 0.4x_{CS} - 0.6x_{DS} \geq 0 \\
 \text{Mixture S Add D :} & -0.25x_{AS} - 0.25x_{BS} - 0.25x_{CS} + 0.75x_{DS} \leq 0 \\
 \text{Mixture E Add C :} & -0.5x_{AE} - 0.5x_{BE} + 0.5x_{CE} - 0.5x_{DE} \geq 0 \\
 \text{Mixture E Add D :} & -0.45x_{AE} - 0.45x_{BE} - 0.45x_{CE} + 0.55x_{DE} \leq 0 \\
 & x_{AD}, x_{AS}, x_{AE}, x_{BD}, x_{BS}, x_{BE}, x_{CD}, x_{CS}, x_{CE}, x_{DD}, x_{DS}, x_{DE} \geq 0.
 \end{array}$$

You are given an Excel spreadsheet that solves this problem. Suppose that the selling prices are changed to \$7.7 per gallon for Deluxe, \$6.8 per gallon for Standard, and \$4.9 per gallon for Economy. In addition, the marketing department estimates that the demand for Deluxe does not exceed 12000 gallons, therefore the corporation does not want to produce more than this quantity for Deluxe. Assume that all other data remains unchanged. Update the Excel spreadsheet and then answer the following questions:

- (a) (10 points) What is the optimal profit and the optimal production-mix?

**Solution.** The optimal profit is \$97801.25 with optimal solution  $x_{AD} = 3675$ ,  $x_{BD} = 4725$ ,  $x_{BS} = 275$ ,  $x_{CD} = 2400$ ,  $x_{CS} = 1100$ ,  $x_{DD} = 1200$ ,  $x_{DS} = 458.333$ , and all remaining variables are zero.

- (b) (5 points) No mixture Economy is produced in the optimal solution. What would the minimum selling price need to be in order for Economy to be worth producing? (Be accurate to within 5 cents).

**Solution.** It is worth producing mixture Economy when its selling price increases to \$5.75.

- (c) (6 points) Suppose that you can increase the selling price of Standard. What would be the increase in the profit if the selling price of mixture Standard per gallon increases to  $6.8 + s$  for  $s = \$ 0.05$ ,  $\$0.1$ , and  $\$0.15$ . (Assume that the selling price of mixtures Deluxe and Economy remain \$7.7 and \$4.9 per gallon, respectively.) The increase is the difference between the new profit and the profit from Part (a).

**Solution.** The increase in the profit for  $s = \$ 0.05$ ,  $\$0.1$ , and  $\$0.15$  is \$ 91.6667, \$183.333, \$275, respectively.

- (d) (8 points) Based on your answer in Part (c), estimate the profit if the selling price of mixture Standard increases to 7.2. What is the formula for the optimum profit if the selling price

of Standard per gallon increased to  $\$6.8 + s$ ? (You may assume that  $s$  is between 0.05 and 0.25).

**Solution.** The increase in the profit at price  $s$  (for small enough  $s$ ) is  $20 \times s \times 91.6667$ .

- (e) (5 points) Based on your formula in part (d), estimate the contribution (the increase in optimal profit) if the selling price of Standard per gallon increases to 7.5. Use Excel solver to see if your estimation is correct.

**Solution.** The increase by the formula is \$12.8334, and the increase in the profit by solving the problem is \$12.8334. The formulae is still correct.

MIT OpenCourseWare  
<http://ocw.mit.edu>

15.053 Optimization Methods in Management Science  
Spring 2013

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.