15.053 February 14, 2013

- Review of Guassian elimination for solving systems of equations
- Introduction to the Simplex Algorithm

Quotes for today

“Any impatient student of mathematics or science or engineering who is irked by having algebraic symbolism thrust upon him should try to get along without it for a week.”

-- Eric Temple Bell

“To become aware of the possibility of the search is to be onto something.”

-- Walker Percy

Overview

- **Review of how to solve systems of equations**
  - **Solving equations using Gaussian elimination.**

- **The simplex algorithm**
  - **a clever search technique**
  - **one of the most important developments in optimization in the last 100 years**

Solving for three variables

E_1	$2\ x_1$	+	$2\ x_2$	+	x_3	=	9
E_2	$2\ x_1$	-	x_2	+	$2\ x_3$	=	6
E_3	x_1	-	x_2	+	$2\ x_3$	=	5

Step 1. Make the coefficients for x_1 in the three equations 1, 0 and 0.

$E_4 = .5\ E_1$	x_1	+	x_2	+	$.5\ x_3$	=	9/2
$E_5 = E_2 – E_1$		-	$3\ x_2$	+	x_3	=	-3
$E_6 = E_3 - .5\ E_1$		-	$2x_2$	+	$1.5\ x_3$	=	1/2

Steps 2 and 3.

E_4	x_1	+	x_2	+	$.5\ x_3$	=	9/2
E_5		-	$3\ x_2$	+	x_3	=	-3
E_6		-	$2x_2$	+	$1.5\ x_3$	=	1/2

$E_7 = E_4 - E_8$	x_1			+	$5\ x_3\ /6$	=	7/2
$E_8 = -\ E_5\ /\ 3$			x_2	-	$x_3\ /\ 3$	=	1
$E_9 = E_6 + 2\ E_8$				+	$5\ x_3\ /6$	=	5/2

$E_{10} = E_7 - 5\ E_{12}\ /\ 6$	x_1					=	1
$E_{11} = E_8 + E_{12}\ /\ 3$			x_2			=	2
$E_{12} = 6\ E_9\ /\ 5$					x_3	=	3

Variation: write variables at the top, and keep track of changes in coefficients.

E_1	$2\ x_1$	+	$2\ x_2$	+	x_3	=	9
E_2	$2\ x_1$	-	x_2	+	$2\ x_3$	=	6
E_3	x_1	-	x_2	+	$2\ x_3$	=	5

	x_1	x_2	x_3		RHS
E_1	2	2	1	=	9
E_2	2	-1	2	=	6
E_3	1	-1	2	=	5

Solve equations as before

	x_1	x_2	x_3		RHS
E_1	2	2	1	=	9
E_2	2	-1	2	=	6
E_3	1	-1	2	=	5

	x_1	x_2	x_3		RHS
$E_4 = .5\ E_1$	1	1	1/2	=	9/2
$E_5 = E_2 - E_1$	0	-3	1	=	-3
$E_6 = E_3 - .5\ E_1$	0	-2	3/2	=	1/2

	x_1	x_2	x_3		RHS
$E_7 = E_4 - E_8$	1	0	5/6	=	7/2
$E_8 = - E_5 / 3$	0	1	-1/3	=	1
$E_9 = E_6 + 2\ E_8$	0	0	5/6	=	5/2

Some notation

	x_1	x_2	x_3		RHS
E_7	1	0	5/6	=	7/2
E_8	0	1	-1/3	=	1
E_9	0	0	5/6	=	5/2

When the equations are written with variables at the top and coefficients are below, it will be called a *tableau*.

1
0
0

and

0
1
0

are *unit vectors* 1_1 and 1_2.

Q1. Suppose that we finish solving the three equations. We have just carried out Steps 1 and 2. After we carry out Step 3, which of the following is not true:

	x_1	x_2	x_3		RHS
E_7	1	0	5/6	=	7/2
E_8	0	1	-1/3	=	1
E_9	0	0	5/6	=	5/2

1. The column for x_3 becomes 1_3.
2. The columns for x_2 and x_3 remain as 1_1 and 1_2.
3. The first equation becomes $x_1 = 7/2$.
4. The third equation gives the solution for x_3.

Pivoting

	x_1	x_2	x_3	x_4		RHS
Row 1	2	2	1	1	=	9
Row 2	$\boxed{2}$	-1	2	0	=	6
Row 3	1	-1	2	1	=	5

To *pivot* on the coefficient in row i and column j is to convert column j into 1_i by

1. multiply row i by a constant
2. add multiples of row i to other rows.

	x_1	x_2	x_3	x_4		RHS
Row 1	0	3	-1	1	=	3
Row 2	$\boxed{1}$	-1/2	1	0	=	3
Row 3	0	-1/2	1	1	=	2

Q2. Suppose that we pivot on the "-1" in Row 1. What is coefficient of x_4 in Row 3 after the pivot?

	x_1	x_2	x_3	x_4		RHS
Row 1	0	3	-1	1	=	3
Row 2	1	-1/2	1	0	=	3
Row 3	0	-1/2	1	1	=	2

A. 0

B. 1

C. 2

D. There is not enough information

Summary of solving equations

x_1	x_2	x_3		RHS
2	2	1	=	9
2	-1	2	=	6
1	-1	2	=	5

To solve for x_1, x_2, and x_3 we

- pivot on row 1, col 1
- pivot on row 2, col 2
- pivot on row 3, col 3

  (assuming the coefficients are non-zero)

This concludes are summary of solving equations.

Linear Programming

- Getting LPs into the correct form for the simplex method
  - changing inequalities (other than non-negativity constraints) to equalities
  - putting the objective function
  - canonical form

- The simplex method, starting from canonical form.

A linear program with inequality constraints.

Consider a linear program in which all variables are non-negative. How can we convert inequality constraints into equality constraints?

$$
\begin{aligned}
\text{max} \quad z = \quad & 3\,x_1 + 2\,x_2 - x_3 + 2\,x_4 \\
& x_1 + 2\,x_2 + x_3 - x_4 \le 5\,; \\
& 2\,x_1 + 4\,x_2 + x_3 + 3\,x_4 \ge 8; \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

We convert a "≤" constraint into a "=" constraint by adding a slack variable, constrained to be ≥ 0.

$$
\begin{aligned}
x_1 + 2\,x_2 + x_3 - x_4 + s_1 = 5\,; \\
s_1 \ge 0
\end{aligned}
$$

Converting a “≥” constraint.

$$2\, x_1 + 4\, x_2 + x_3 + 3\, x_4 \quad \geq \; 8;$$

We convert a “≥” constraint into a “=“ constraint by subtracting a surplus variable, constrained to be ≥ 0.

$$2\, x_1 + 4\, x_2 + x_3 + 3\, x_4 \; - s_2 = \; 8\;;$$

$$s_2 \geq 0$$

Whenever we transform a new constraint, we create a new variable. There is only one equality constraint for each slack variable and for each surplus variable.

Creating an LP tableau from an LP

Assumptions:

- **All variables are nonnegative**
- **All other constraints are "=" constraints.**

$$\begin{aligned} \max \quad z = \; & 3x_1 + 2x_2 - x_3 + 2x_4 \\ & x_1 + 2x_2 + x_3 - x_4 + s_1 = 5; \\ & 2x_1 + 4x_2 + x_3 + 3x_4 - s_2 = 8; \\ & x_1, x_2, x_3, x_4, s_1, s_2 \geq 0 \end{aligned}$$

Question: what variables should we include?

what about the objective function?

An LP tableau

$$\max \quad z = 3x_1 + 2x_2 - x_3 + 2x_4$$

$$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5;$$

$$2x_1 + 4x_2 + x_3 + 3x_4 - s_2 = 8;$$

$$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0$$

$$\boxed{-z + 3x_1 + 2x_2 - x_3 + 2x_4 = 0}$$

-z	x_1	x_2	x_3	x_4	s_1	s_2		RHS
1	3	2	-1	2	0	0	=	0
0	1	2	1	-1	1	0	=	5
0	2	4	1	3	0	-1	=	8

The simplex method begins with an LP in canonical form

$-z$	x_1	x_2	x_3	x_4	s_1	s_2		RHS
1	3	2	-1	2	0	0	=	0
0	1	2	1	-1	1	0	=	5
0	2	4	1	3	0	-1	=	8

An LP tableau is in canonical form if all of the following are true.

1. **All decision variables are non-negative (except for $-z$)**
2. **All (other) constraints are equality constraints.**
3. **The RHS is non-negative (except for cost row)**
4. **For each row i, there is a column equal to 1_i.**

An LP in canonical form

$-z$	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	3	-2	-1	0	1	0	=	0
0	1	-2	1	1	-1	0	=	5
0	2	-4	-1	0	2	1	=	1

Our checklist from the previous slide

1. All decision variables are non-negative (except for–
2. All (other) constraints are equality constraints.
3. The RHS is non-negative (except for cost row)
4. For each row i, there is a column equal to 1_i.

On the row with the objective function

$-z$	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	3	-2	-1	0	1	0	=	0

We will refer to the row with the objective function as the "z-row" It's a term that is used only in 15.053 and 15.058.

Professor Orlin accidentally referred to this row as the "z-row" a decade ago, and found it amusing at that time because it sounds the same as 0.

He still uses the term.

Q3. Consider the tableau below, where a, b, c, and c are unknown. Under what conditions is the tableau in canonical form? Select the best answer.

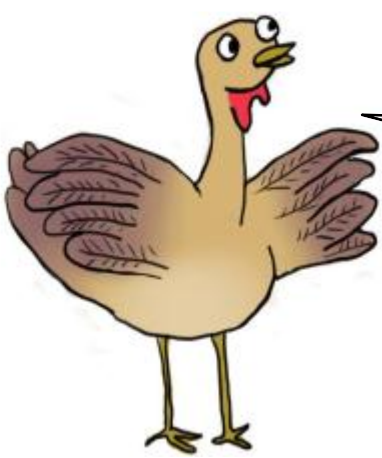
-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	3	-4	-1	0	1	0	=	a
0	1	-2	1	1	-1	0	=	5
0	2	-4	-1	b	2	1	=	c

1. $a \geq 0$
   $b = 0$,
   $c \geq 0$.
2. $a \leq 0$
   $b = 0$,
   $c > 0$.
3. $b = 0$,
   $c \geq 0$.
4. $b = 0$,
   $c > 0$

1. All decision variables are non-negative (except for–z)
2. All (other) constraints are equality constraints.
3. The RHS is non-negative (except for cost row)
4. For each row i, there is a column equal to 1_i.

The simplex method will start with a tableau in canonical form. Is it easy to put a linear program into canonical form?

It's pretty easy to satisfy conditions 1 to 3. It's called putting an LP into standard form. Condition 4 is tricky. We'll explain how to do it next lecture. For now, I ask you and the students to accept that we start in canonical form.



OK. For now.

Mental Break

Basic variables, non-basic variables, and basic feasible solutions.

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	3	-2	-1	0	1	0	=	0
0	1	-2	1	1	-1	0	=	5
0	2	-4	-1	0	2	1	=	1

The basic variables are the variables corresponding to the identity matrix. $\{-z, x_4, x_6\}$.

The nonbasic variables are the remaining variables. $\{x_1, x_2, x_3, x_5\}$

The basic feasible solution is the unique solution obtained by setting the non-basic variables to 0.

$z = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 5$, $x_5 = 0$, $x_6 = 1$.

Same problem, different basic variables.

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	3	-2	-1	0	1	0	=	0
0	1	-2	1	1	-1	0	=	5
0	2	-4	-1	0	2	1	=	1

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	4	-4	0	1	0	0	=	5
0	1	-2	1	1	-1	0	=	5
0	3	-6	0	1	1	1	=	6

What are the basic variables? $\{-z, x_3, x_6\}$.

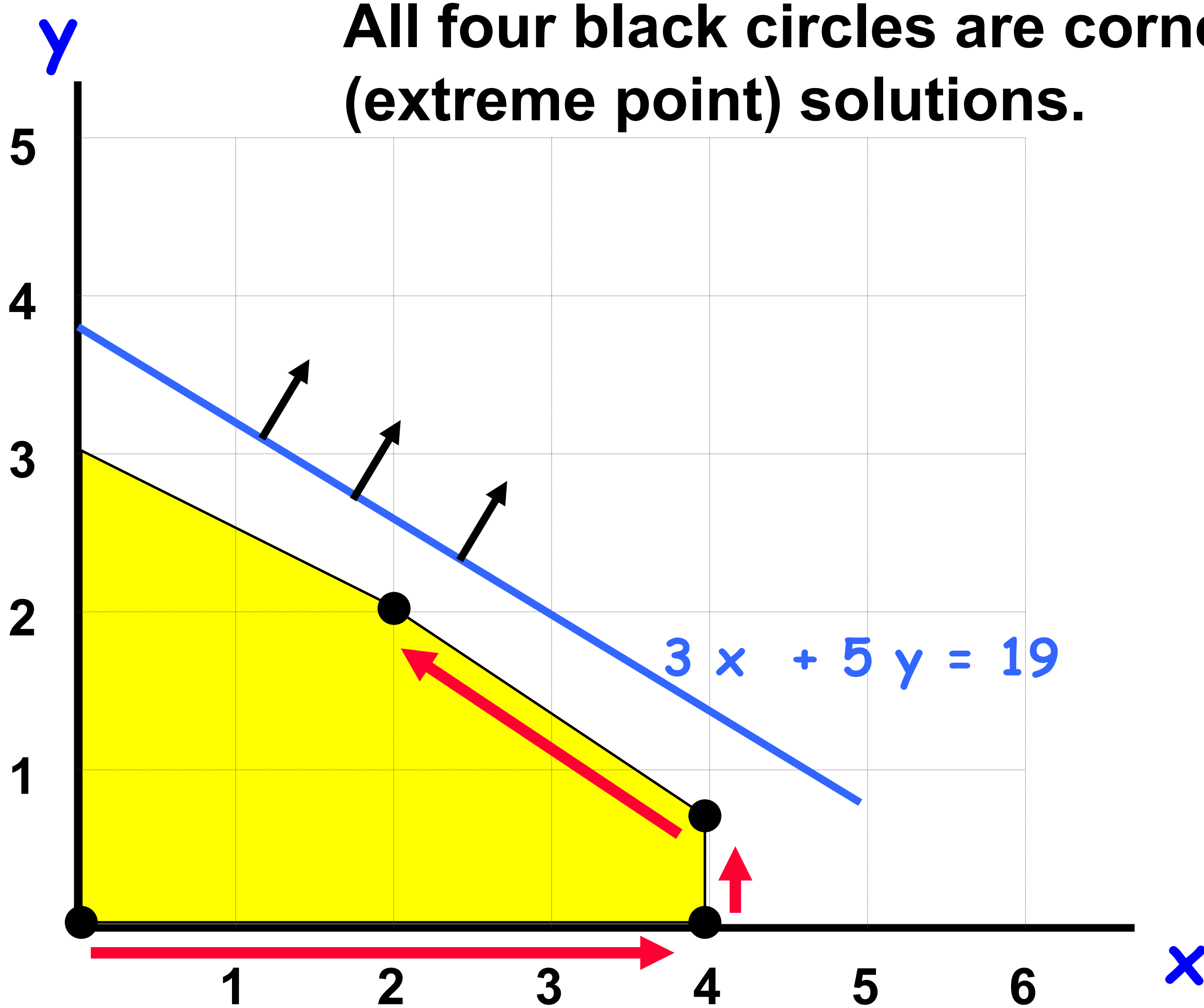
What are the nonbasic variables? $\{x_1, x_2, x_4, x_5\}$

What is the basic feasible solution?

$z = -5$, $x_1 = 0$, $x_2 = 0$, $x_3 = 5$, $x_4 = 0$, $x_5 = 0$, $x_6 = 6$.

A basic feasible solution is a corner point solution.

All four black circles are corner point (extreme point) solutions.



A warm exercise about optimality conditions.

Q4. What is the optimal objective value for the following linear program.

maximize $z = -3\,x_1 - 4\,x_2 - 0\,x_3 + 13$

subject to $x_1, x_2, x_3 \geq 0$

A. 0

B. 13

C. 20

D. There is not enough information

Optimality conditions for a maximization problem

Optimality Condition. A basic feasible solution is optimal if every coefficient in the z-row is non-positive.

Basic Var	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-13	0	0	-1	=	-17
x_3	0	0	2	1	0	2	=	4
x_4	0	0	-1	0	1	-2	=	1
x_1	0	1	6	0	0	1	=	3

	z	x_1	x_2	x_3	x_4	x_5
BFS	17	3	0	4	1	0

Objective: $z = 0\ x_1 - 13\ x_2 + 0\ x_3 + 0\ x_4 – x_5 + 17.$

There can be no solution with $x \geq 0$ that has value > 17

Some LP notation

-z	x_1	x_2		x_6		RHS
1	c_1	c_2	…	c_n	=	$-z_0$
0	a_{11}	a_{12}	…	a_{1n}	=	b_1
…	…	…	……			
0	a_{m1}	a_{m2}	…	a_{mn}	=	b_m

The initial tableau for an LP

c_i is the cost coefficient for variable x_i.

-z	x_1	x_2		x_6		RHS
1	$\bar{c}_1$	$\bar{c}_2$	…	$\bar{c}_n$	=	$\overline{-z}_0$
0	$\bar{a}_{11}$	$\bar{a}_{12}$	…	$\bar{a}_{1n}$	=	$\bar{b}_1$
…	…	…	……			
0	$\bar{a}_{m1}$	$\bar{a}_{m2}$	…	$\bar{a}_{mn}$	=	$\bar{b}_m$

$\bar{c}_i$ is the reduced cost for variable x_i.

The tableau for the same LP after pivoting

Optimality conditions for a maximization problem

Optimality Condition. A basic feasible solution is optimal if the reduced cost of every variable (except z) is non-positive.

Basic Var	-z	x_1	x_2	x_3	x_4	x_5		RHS
-z	1	0	-13	0	0	-1	=	-17
x_3	0	0	2	1	0	2	=	4
x_4	0	0	-1	0	1	-2	=	1
x_1	0	1	6	0	0	1	=	3

How to obtain a better solution if the bfs is not optimal.

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	4	-4	0	-1	0	0	=	- 3
0	1	-2	1	1	-1	0	=	5
0	3	-6	0	1	1	1	=	6

$z = 4x_1 - 4x_2 - x_4 + 3$

Choose i so that $\bar{c}_i > 0$. (choose i = 1)

- Note: x_i is a nonbasic variable

Increase x_1.

Avoid increasing x_2, x_4, x_5. (Do not change the value of any of the other nonbasic variables).

Finding a solution with higher profit.

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	4	-4	0	-1	0	0	=	- 3
0	1	-2	1	1	-1	0	=	5
0	3	-6	0	1	1	1	=	6

$$z = 4\,x_1 - 4\,x_2 - x_4 + 3$$

Increase x_1. (x_1 is called the *entering variable*.) Keep other non-basic variables at 0 (x_2 and x_4 and x_5). Adjust the basic variables x_3 and x_6 to maintain feasibility.

$$\begin{aligned} -z + 4\,x_1 \quad\quad &= -3 \\ x_1 + x_3 \quad &= 5 \\ 3x_1 + \quad x_6 &= 6 \end{aligned}$$

$$\begin{aligned} z &= 3 + 4\,x_1 \\ x_3 &= 5 - x_1 \\ x_6 &= 6 - 3\,x_1 \end{aligned}$$

Moving along an edge: The Δ-Method

$$z = 3 + 4x_1$$
$$x_3 = 5 - x_1$$
$$x_6 = 6 - 3x_1$$

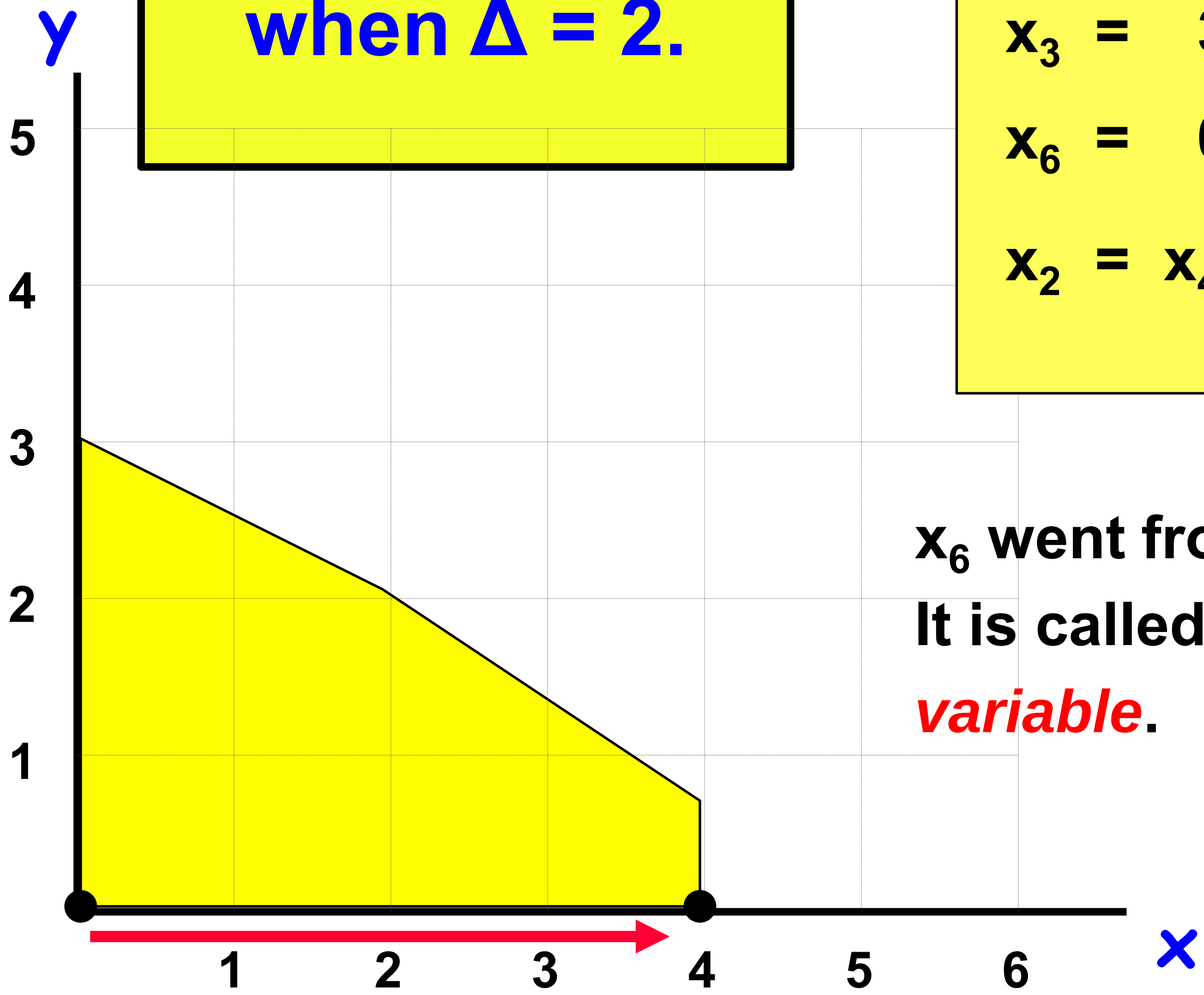
To express the edge, write all variables in terms of a single parameter Δ.

$$x_1 = \Delta$$
$$z = 3 + 4\Delta$$
$$x_3 = 5 - \Delta$$
$$x_6 = 6 - 3\Delta$$
$$x_2 = x_4 = x_5 = 0$$

The *edge* consists of all vectors x, z that can be formed on the left for $0 \le \Delta \le 2$.

Why are the bounds 0 and 2?

The next corner point occurs when $\Delta = 2$.



$x_1 = 2$

$z = 11$

$x_3 = 3$

$x_6 = 0$

$x_2 = x_4 = x_5 = 0$

x_6 went from positive to 0. It is called the *exiting variable*.

Next steps

- How to recognize unboundedness
- A shortcut that permits one to pivot to the next basic feasible solution (corner point solution)
- But first, a quick review

-z	x_1	x_2	x_3	x_4	x_5		RHS
1	0	-2	0	0	+1	=	-6
0	0	2	1	0	2	=	4
0	0	-1	0	1	-2	=	2
0	1	6	0	0	1	=	3

What is the basic feasible solution?

What is the edge that corresponds to increasing the entering variable?

What is the entering variable?

What is the next basic feasible solution? What is the exiting variable?

Unboundedness

Theorem. ***If the column coefficients (except for the z-row) of the entering variable are non-positive, then the objective value is unbounded from above.***

-z	x_1	x_2	x_3	x_4	x_5	x_6		RHS
1	4	2	0	1	0	0	=	5
0	1	-1	1	1	-1	0	=	5
0	3	0	0	1	1	1	=	6

Suppose that x_2 enters.

Let $x_2 = \Delta$.

$x_1 = x_4 = x_5 = 0$

$z = 2\Delta - 5$

$x_3 = \Delta + 5$

$x_6 = 6$

As Δ increases, z increases.

There is no upper bound on Δ.

The Min Ratio Rule

$-z$	x_1	x_2	x_3	x_4	x_5		RHS	Ratio of RHS to Col
1	0	-2	0	0	+1	=	-6	
0	0	2	1	0	2	=	4	4/2
0	0	-1	0	1	-2	=	2	coef ≤ 0
0	1	6	0	0	1	=	3	3/1

$x_1 = 3 - \Delta$;

$x_2 = 0$;

$x_3 = 4 - 2\Delta$;

$x_4 = 2 + 2\Delta$;

$x_5 = \Delta$;

$z = 6 + \Delta$;

$0 \le \Delta \le 2$

$$\Delta_{max} = \min \frac{\text{RHS coef}}{\text{col. coef}}$$

s.t. col. coef > 0

The **exiting variable** is the basic variable in the row with the min ratio.

The simplex pivot rule: pivot on the column of the entering variable and the row which gave the min ratio.

$-z$	x_1	x_2	x_3	x_4	x_5		RHS
1	0	-2	0	0	+1	=	-6
0	0	2	1	0	2	=	4
0	0	-1	0	1	-2	=	2
0	1	6	0	0	1	=	3

$-z$	x_1	x_2	x_3	x_4	x_5		RHS
1	0	-3	-0.5	0	0	=	-8
0	0	1	0.5	0	1	=	2
0	0	1	1	1	0	=	6
0	1	5	-0.5	0	0	=	1

The entering variable is x_2. What is the leaving variable?

$-z$	x_1	x_2	x_3	x_4	x_5		RHS
1	0	+2	0	0	-1	=	-2
0	0	2	1	0	2	=	4
0	0	-1	0	1	-2	=	1
0	1	6	0	0	1	=	3

1. x_1
2. x_3
3. x_4
4. $-z$

Summary for maximization.

1. Find a variable x_s so that its cost coefficient is positive.
2. Let $x_s = \Delta$.
3. Adjust the basic variables as a function of Δ. Choose Δ maximal.
4. Arrive at a new corner point or else increase Δ infinitely and prove that the max objective value is unbounded from above.

Next Lecture

- Review the simplex method
- Show how to obtain an initial bfs
- Prove finiteness (under some assumptions)