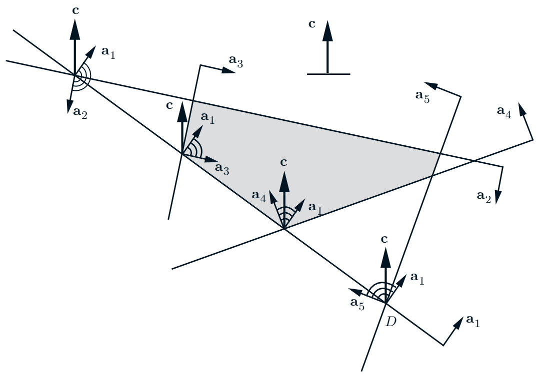
15.093 Optimization Methods

Lecture 6: Duality Theory II



1 Outline

SLIDE 1

- Geometry of duality
- The dual simplex algorithm
- Farkas lemma
- Duality as a proof technique

2 The Geometry of Duality

SLIDE 2

$$\begin{array}{rll} \min & \boldsymbol{c}'\boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \geq b_i, & i = 1, \ldots, m \end{array}$$

$$\begin{array}{rl} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} p_i \boldsymbol{a}_i = \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

3 Dual Simplex Algorithm

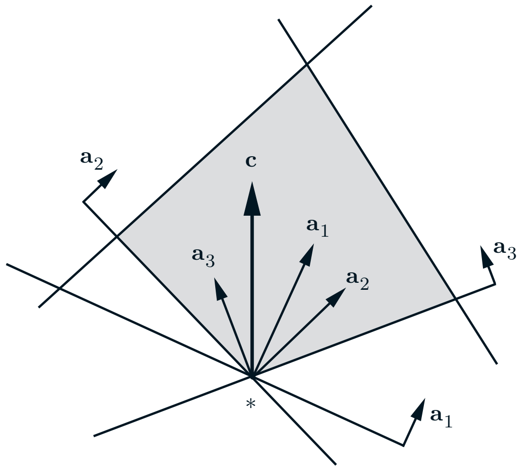
3.1 Motivation

SLIDE 3

- In simplex method $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- **Primal optimality condition**

$$\boldsymbol{c}' - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$$

  same as **dual feasibility**



- Simplex is a **primal algorithm**: maintains **primal feasibility** and works towards **dual feasibility**
- **Dual algorithm**: maintains **dual feasibility** and works towards **primal feasibility**

SLIDE 4

$-\boldsymbol{c}_B'\boldsymbol{x}_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_1$	$\ldots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

- Do not require $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- Require $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$ (dual feasibility)
- Dual cost is
$$\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B'\boldsymbol{x}_B$$
- If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$ then both dual feasibility and primal feasibility, and also same cost $\Rightarrow$ **optimality**
- Otherwise, change basis

3.2 An iteration

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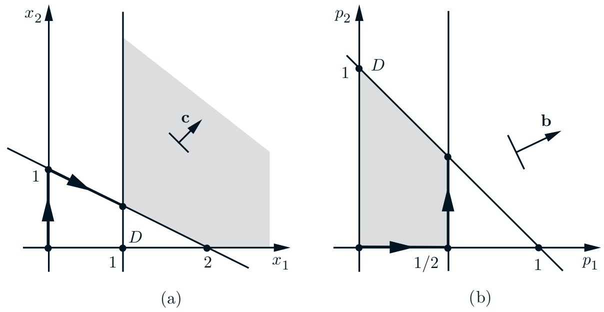
1. Start with basis matrix $\boldsymbol{B}$ and all reduced costs ≥ 0.
2. If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$ optimal solution found; else, choose l s.t. $x_{B(l)} < 0$.



3. Consider the lth row (pivot row) $x_{B(l)}, v_1, \dots, v_n$. If $\forall i \; v_i \geq 0$ then dual optimal cost $= +\infty$ and algorithm terminates.

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4. Else, let j s.t.

$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i | v_i < 0\}} \frac{\bar{c}_i}{|v_i|}$$

5. Pivot element v_j: $\boldsymbol{A}_j$ enters the basis and $\boldsymbol{A}_{B(l)}$ exits.

3.3 An example

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$$\begin{array}{rl} \min & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{rl} \min & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 - x_3 = 2 \\ & x_1 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array} \qquad \begin{array}{rl} \max & 2p_1 + p_2 \\ \text{s.t.} & p_1 + p_2 \leq 1 \\ & 2p_1 \leq 1 \\ & p_1, p_2 \geq 0 \end{array}$$

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		x_1	x_2	x_3	x_4
	0	1	1	0	0
$x_3 =$	-2	-1	-2^*	1	0
$x_4 =$	-1	-1	0	0	1

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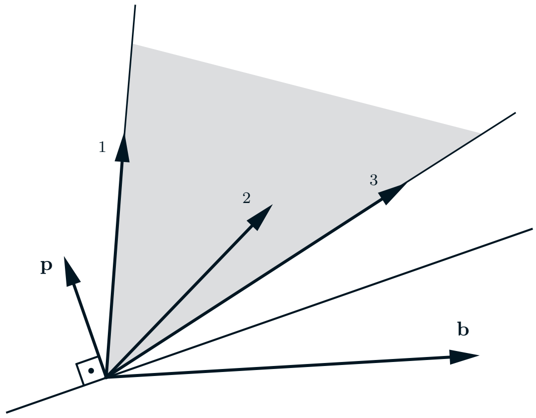
		x_1	x_2	x_3	x_4
	-1	1/2	0	1/2	0
$x_2 =$	1	1/2	1	$-1/2$	0
$x_4 =$	-1	-1^*	0	0	1



		x_1	x_2	x_3	x_4
	$-3/2$	0	0	$1/2$	$1/2$
$x_2 =$	$1/2$	0	1	$-1/2$	$1/2$
$x_1 =$	1	1	0	0	-1

4 Duality as a proof method

4.1 Farkas lemma

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Theorem:
Exactly one of the following two alternatives hold:

1. $\exists \boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
2. $\exists \boldsymbol{p}$ s.t. $\boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$ and $\boldsymbol{p}'\boldsymbol{b} < 0$.

4.1.1 Proof

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"$\Rightarrow$" If $\exists \boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and if $\boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$, then $\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{p}'\boldsymbol{A}\boldsymbol{x} \geq 0$
"$\Leftarrow$" Assume there is no $\boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{array}{llrcl} (P) & \max & \mathbf{0}'\boldsymbol{x} & & \\ & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & & \boldsymbol{x} & \geq & \mathbf{0} \end{array} \qquad \begin{array}{lll} (D) & \min & \boldsymbol{p}'\boldsymbol{b} \\ & \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}' \end{array}$$

(P) infeasible $\Rightarrow$ (D) either unbounded or infeasible
Since $\boldsymbol{p} = \mathbf{0}$ is feasible $\Rightarrow$ (D) unbounded
$\Rightarrow \exists \boldsymbol{p}: \quad \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$ and $\boldsymbol{p}'\boldsymbol{b} < 0$