15.093 Optimization Methods

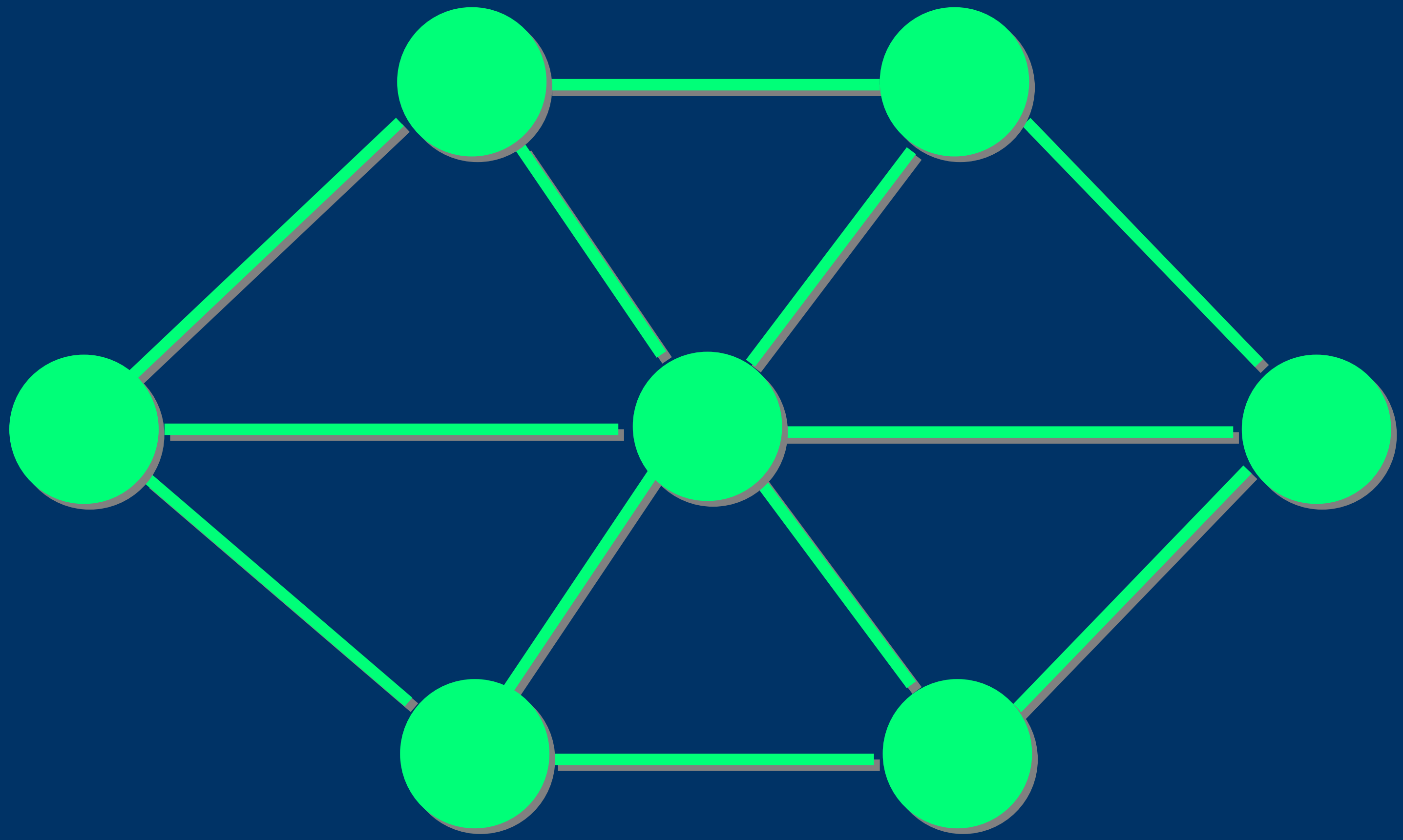
Lecture10: Network Optimization
Introduction and Applications

Network Optimization

What is a network?

Networks

Formally



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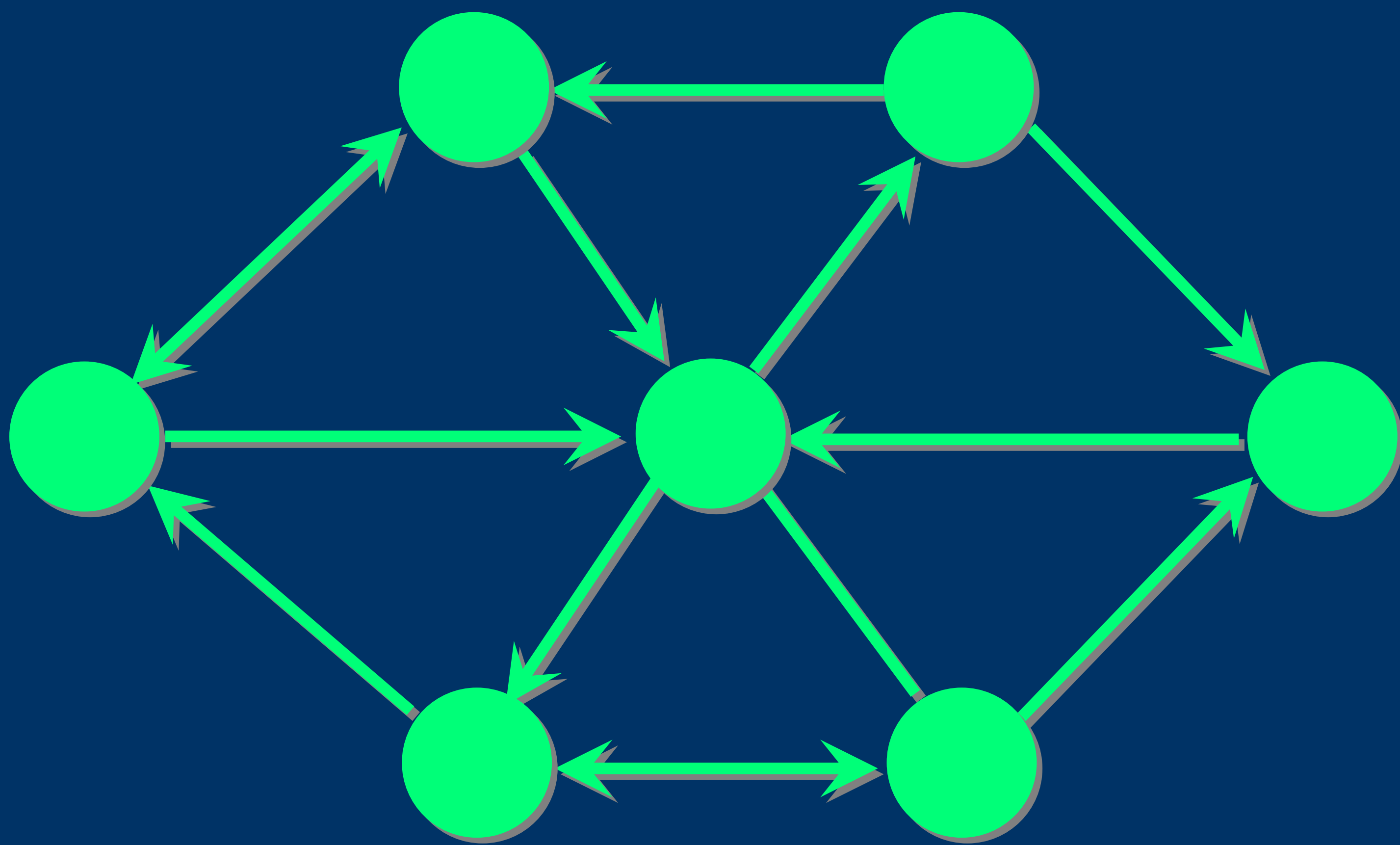
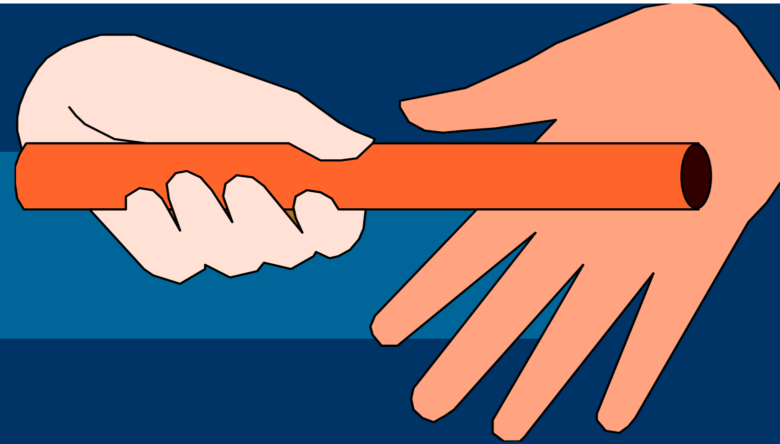


Common Thrust



Move some entity (electricity, a consumer product, a person, a vehicle, a message, . . .) from one point to another in the underlying network, as efficiently as possible.

Lecture 1: Learn how to model application settings as network flow problems.

Lecture 2: Study ways to solve the resulting models.

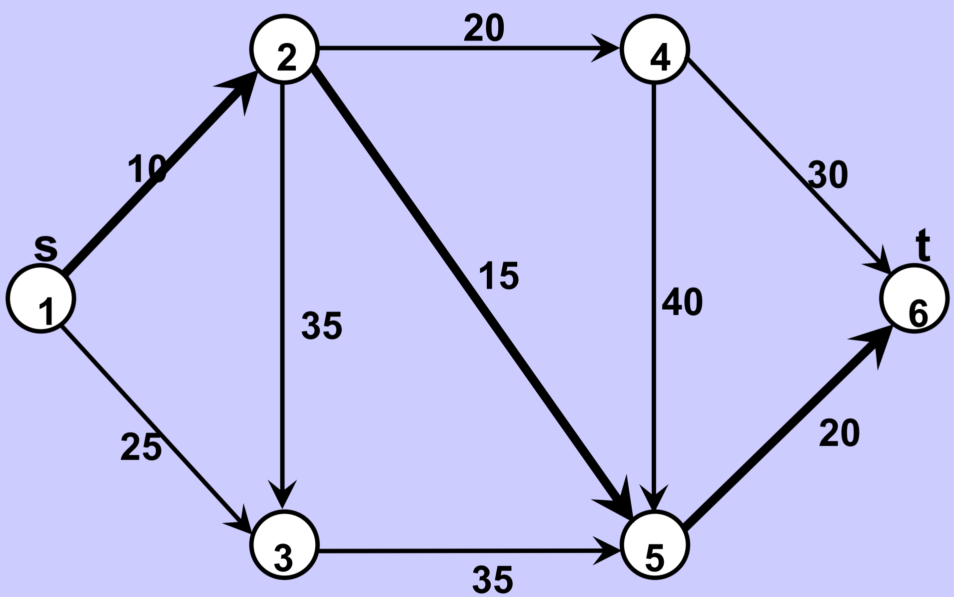
Outline

- Network flow problems
- Applications of the shortest path problem
- Applications of the maximum flow problem
- Applications of the minimum cost flow problem
- Extended models

Shortest Path

Description

Identify a shortest path from a given source node to a given sink node.

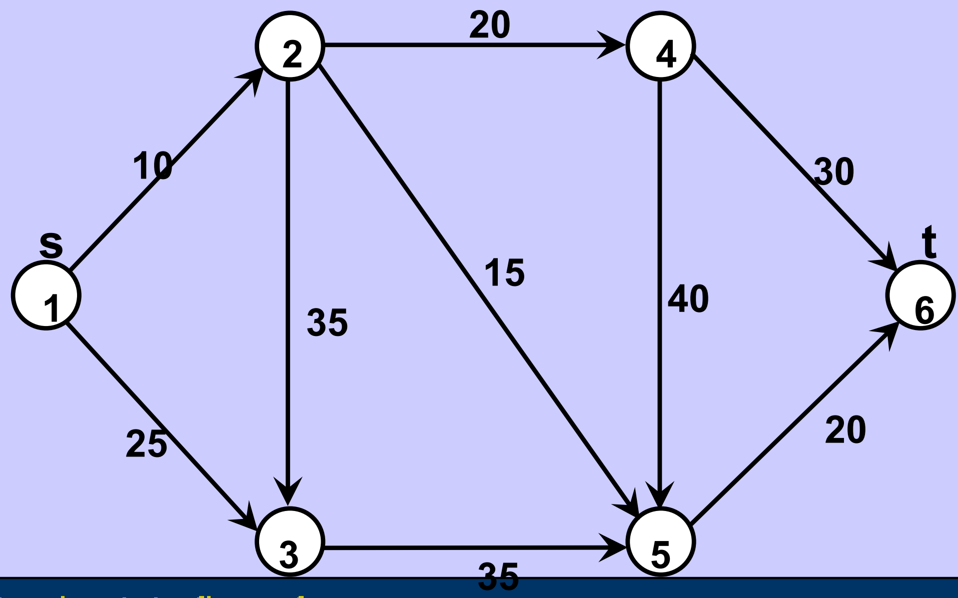


- Finding a path of minimum length.
- Finding a path taking minimum time.
- Finding a path of maximum reliability.

Maximum Flow

Description

Determine the maximum flow that can be sent from a given source node to a sink node in a capacitated network.



Determining maximum steady-state flow of

- petroleum products in a pipeline network,
- cars in a road network,
- messages in a telecommunication network,
- electricity in an electrical network.

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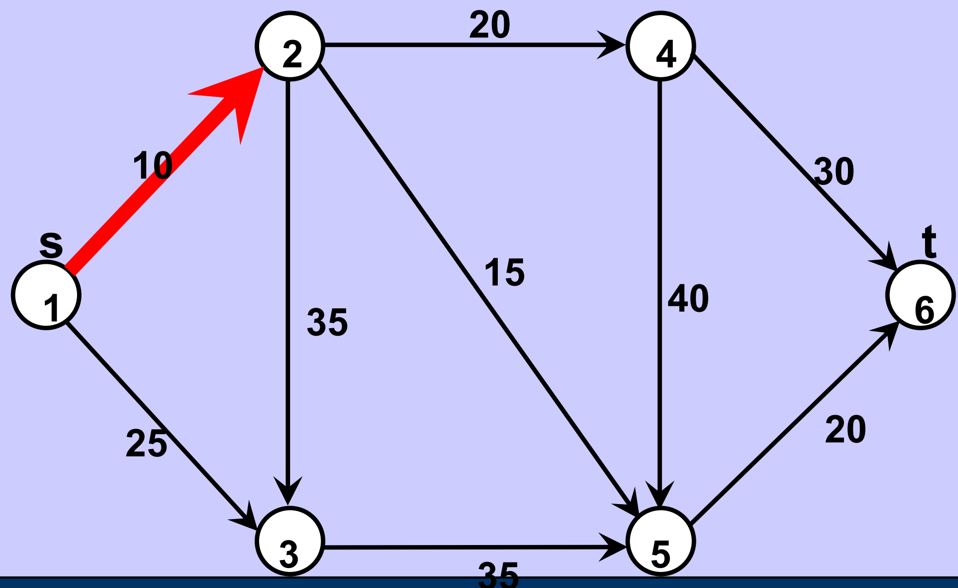

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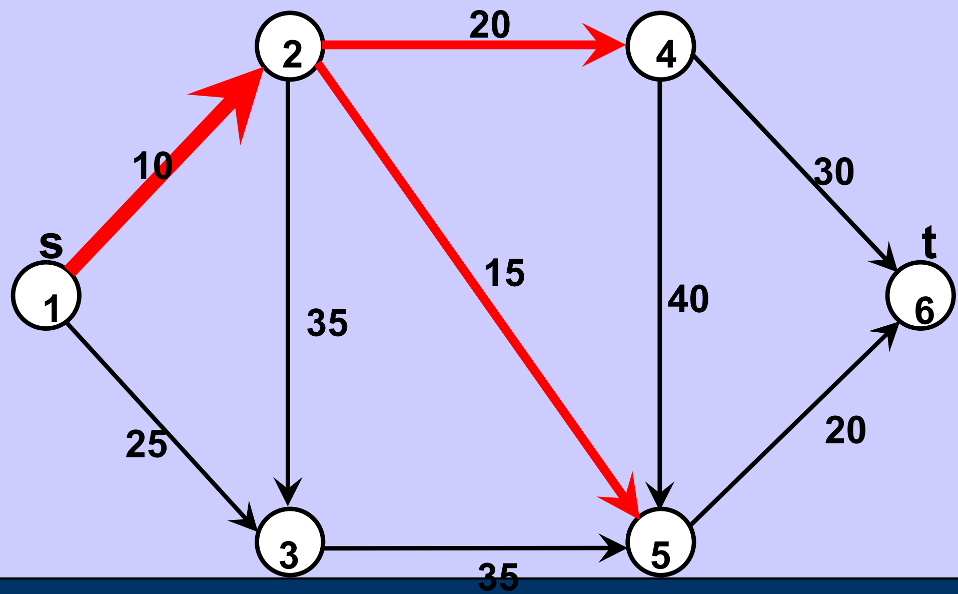

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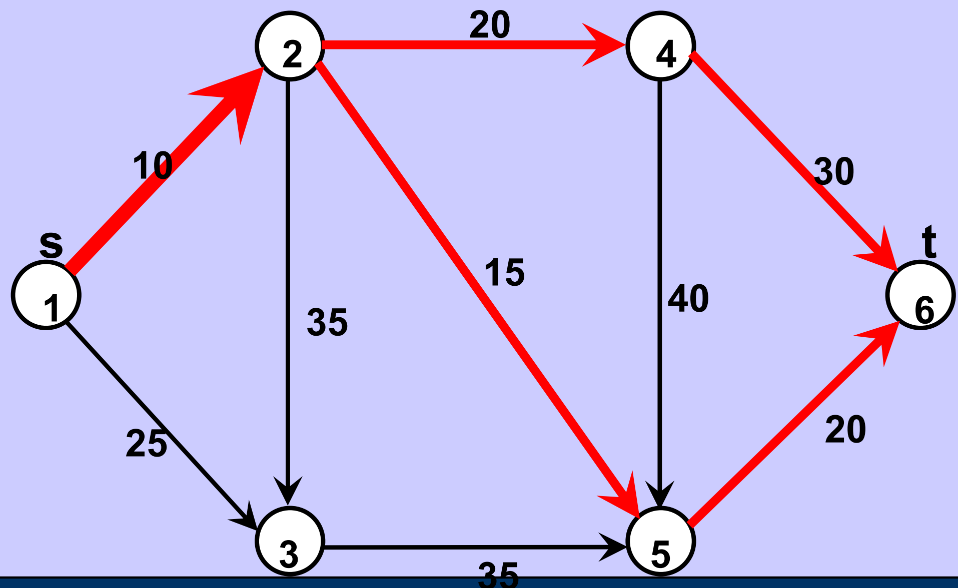

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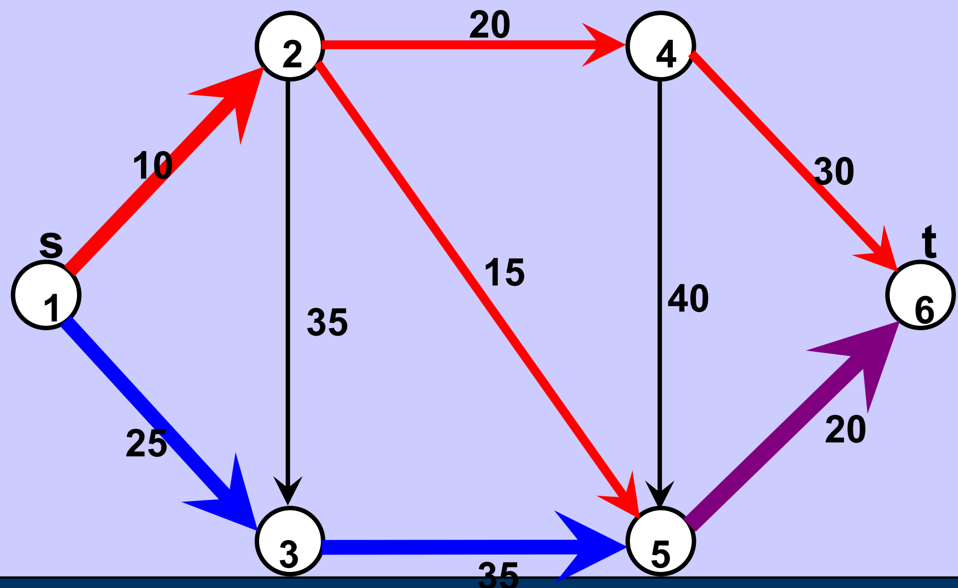

Determining maximum steady-state flow of

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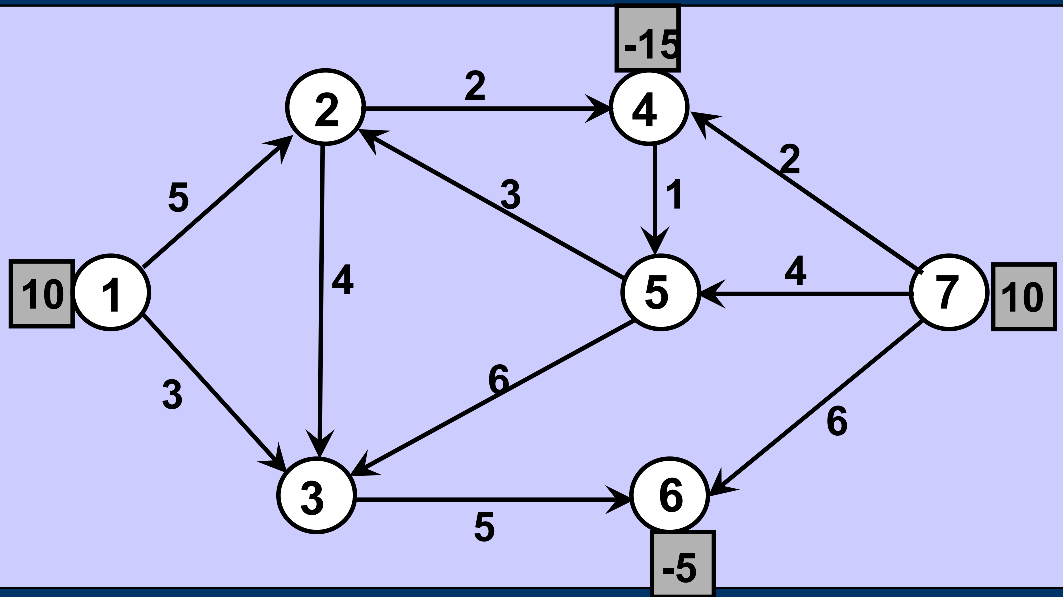
Min-Cost Flow

Description

Determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes. Arcs have capacities and cost associated with them.



- Distribution of products.
- Flow of items in a production line.
- Routing of cars through street networks.
- Routing of telephone calls.

In LOP Form

Min-Cost Flow

- Network $G = (N, A)$.
- Arc costs $c : A \to \mathbb{Z}$.
- Arc capacities $u : A \to \mathbb{N}$.
- Node balances $b : N \to \mathbb{Z}$.

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} & = & b_i & \text{for all } i \in N \\
& x_{ij} & \leq & u_{ij} & \text{for all } (i,j) \in A \\
& x_{ij} & \geq & 0 & \text{for all } (i,j) \in A
\end{array}
$$

Shortest Path

Interword Spacing in $\LaTeX$

The spacing between words and characters is normally set automatically by LaTeX. Interword spacing within one line is uniform. LaTeX also attempts to keep the word spacing for different lines as nearly the same as possible.

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Shortest Path

Interword Spacing in LaTeX (2)

- The paragraph consists of n words, indexed by $1, 2, \ldots, n$.
- c_{ij} is the attractiveness of a line if it begins with i and ends with $j-1$.
- (LaTeX uses a formula to compute the value of each c_{ij}.)

For instance,

$$c_{12} = -10,000 \qquad c_{13} = -1,000$$

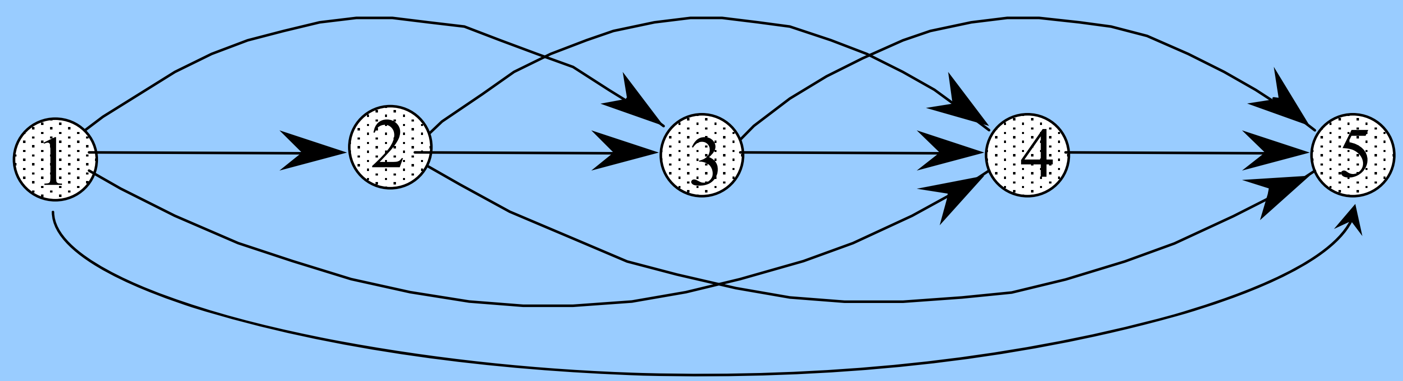
$$c_{14} = 100 \qquad c_{1,37} = -100,000$$

$$\ldots$$

Interword Spacing in LaTeX (3)

Theorem 1

The problem of decomposing a paragraph into several lines of text to maximize total attractiveness can be formulated as a shortest path problem.



Shortest Path

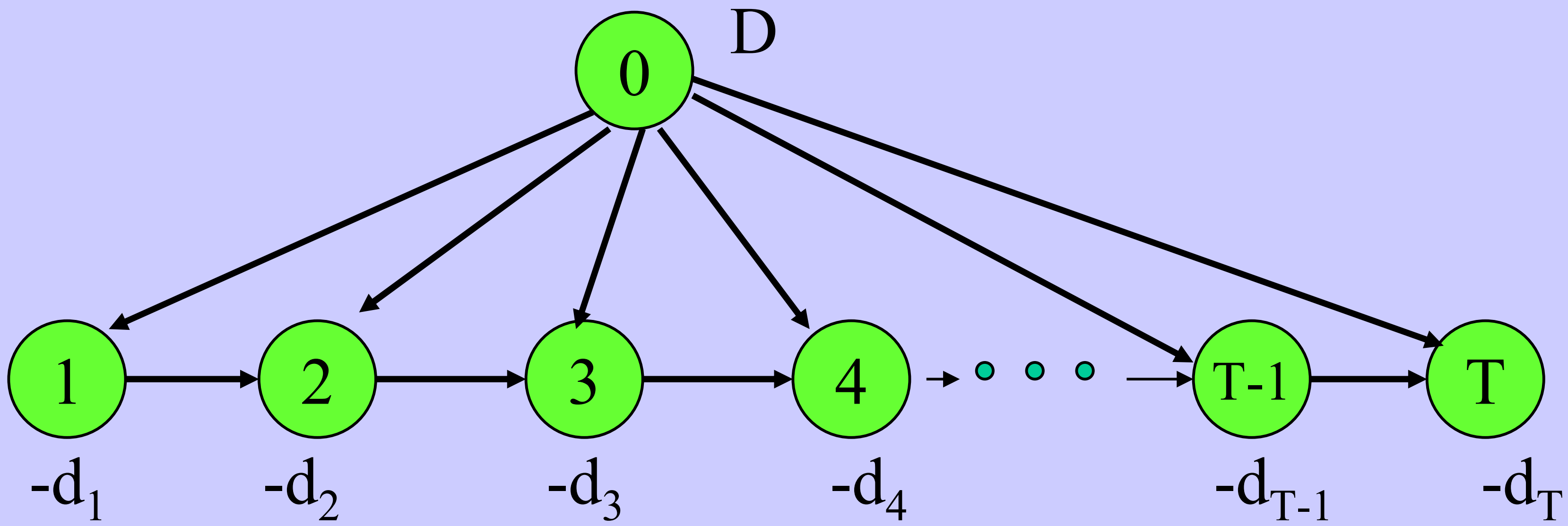
Dynamic Lot Sizing

- T periods of demand for a product. The demand is $d_t > 0$ in period t.
- Let x_t be the production in period t (to be determined).
- Production cost $f_t(x_t) = a_t + b_t x_t$.
- Let I_t be the inventory carried from period t to period $t + 1$.
- $h_t I_t$ linear cost of carrying inventory.

What is the minimum cost way of meeting demand?

Shortest Path

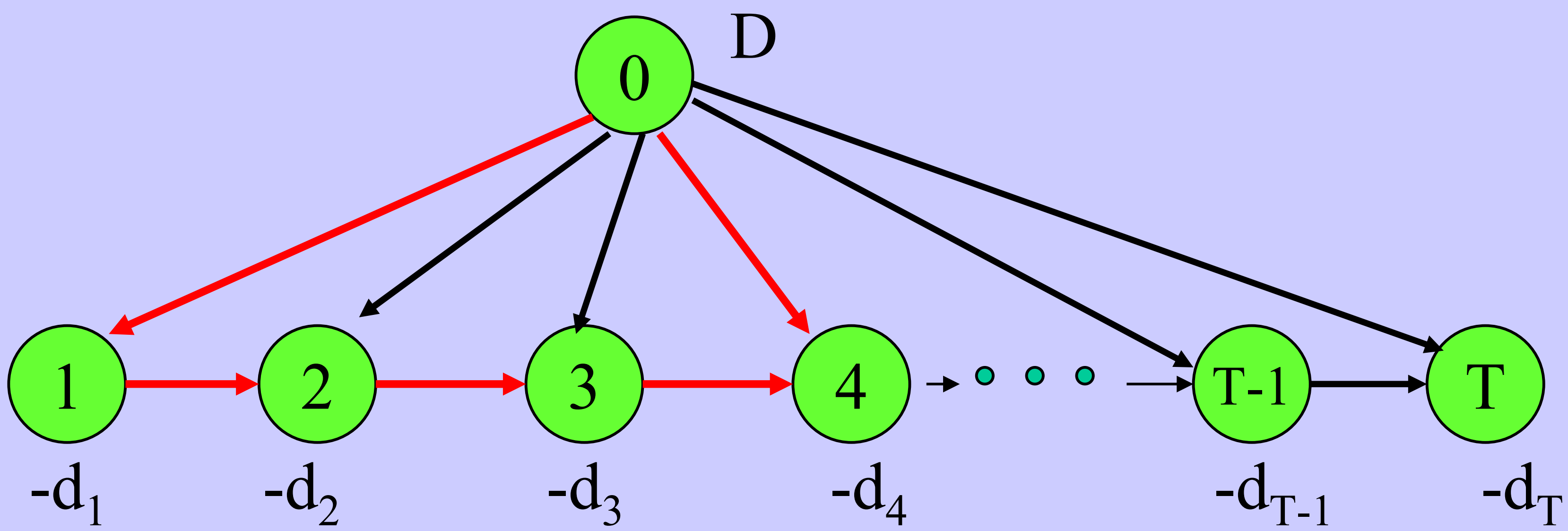
Dynamic Lot Sizing (2)



Lemma 1 *There is exactly one arc with positive flow directed into each node* $t > 0$.

Dynamic Lot Sizing (2)

Shortest Path

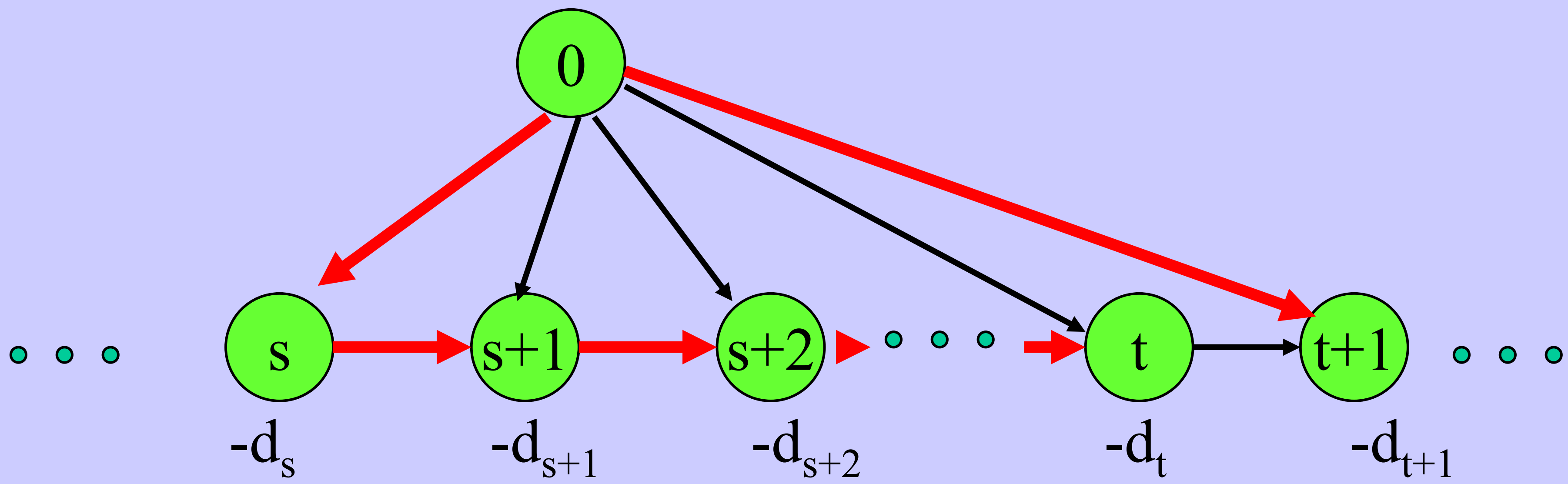


Lemma 1 *There is exactly one arc with positive flow directed into each node* $t > 0$*.*

Dynamic Lot Sizing (3)

Shortest Path

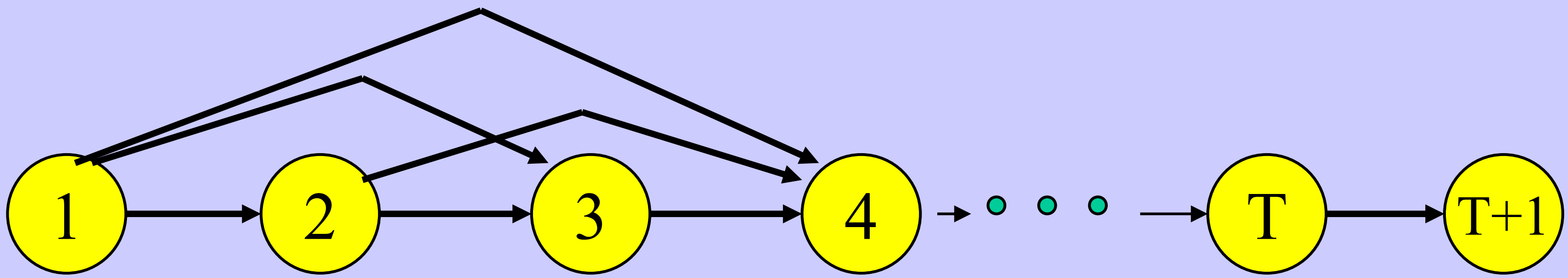
Corollary 1 *Production in period s satisfies demands exactly in periods $s, s+1, \ldots, t$, for some t.*



Dynamic Lot Sizing (4)

Shortest Path

Theorem 2 *The optimal production and inventory schedule can be determined by solving a shortest path problem.*



Let c_{st} be the cost of producing in period s to meet demands in periods $s, s+1, \ldots, t-1$ (including cost of inventory).

Maximum Flow

Baseball Elimination

http://riot.ieor.berkeley.edu/~baseball/

Maximum Flow

Baseball Elimination (2)

Team 0 is "our" team (e.g., Boston Red Sox).

There are n other teams.

$w_i =$ number of wins team i has so far.

$g_{ij} =$ number of games left between teams i and j.

Our team is eliminated if, for all possible ways of playing out the rest of the season, there is always another team that ends up with more wins than our team.

Maximum Flow

Baseball Elimination (3)

Nodes for each team $i \neq 0$, for every pair $\{i, j\}$ of teams $(i, j \neq 0)$, source s, and sink t.

Arcs:

- $(i, \{i, j\})$ with capacity $+\infty$.
- (s, i) with capacity $w_0 + \sum_j g_{0j} - w_i$.
- $(\{i, j\}, t)$ with capacity g_{ij}.

Interpretation:

Flow on arc $(i, \{i, j\}) =$ # of remaining games with j that i wins.

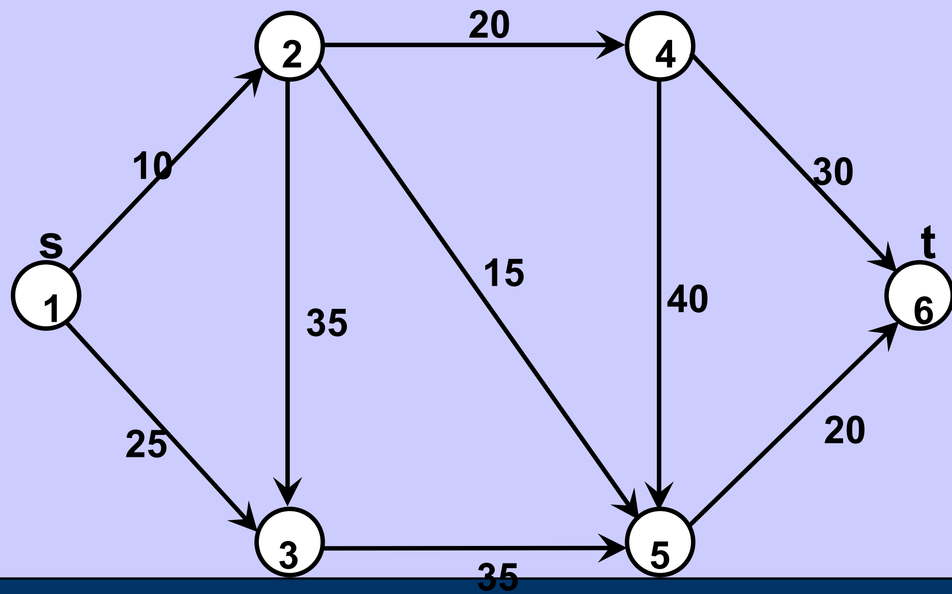
Baseball Elimination (4)

Theorem 3 *There is a flow saturating* t *if and only if there is a way to play out the season where Team* 0 *is not eliminated.*

Maximum Flow

Max-Flow vs. Min-Cut

What is the value of the max flow in this network?



Maximum Flow

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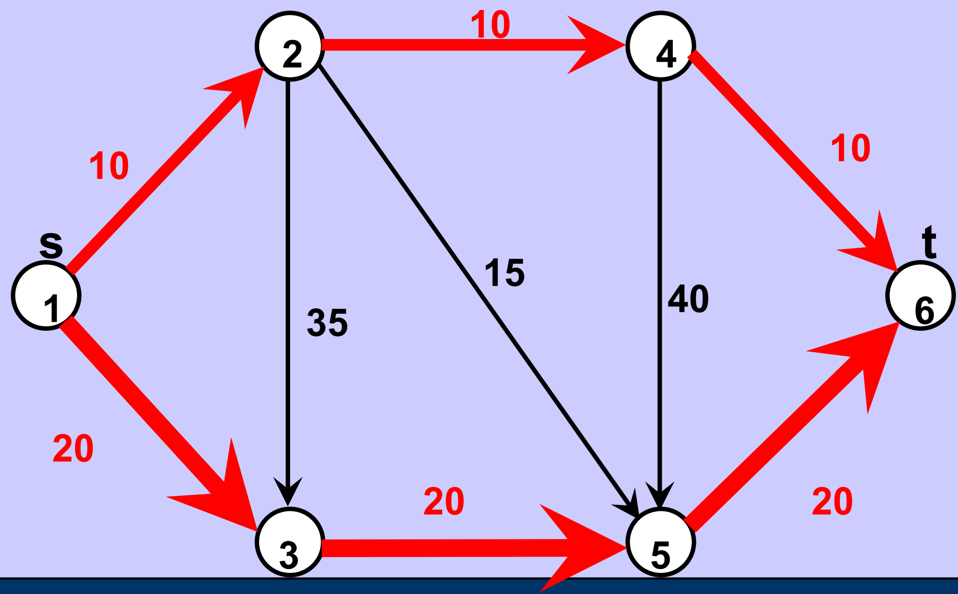

Maximum Flow

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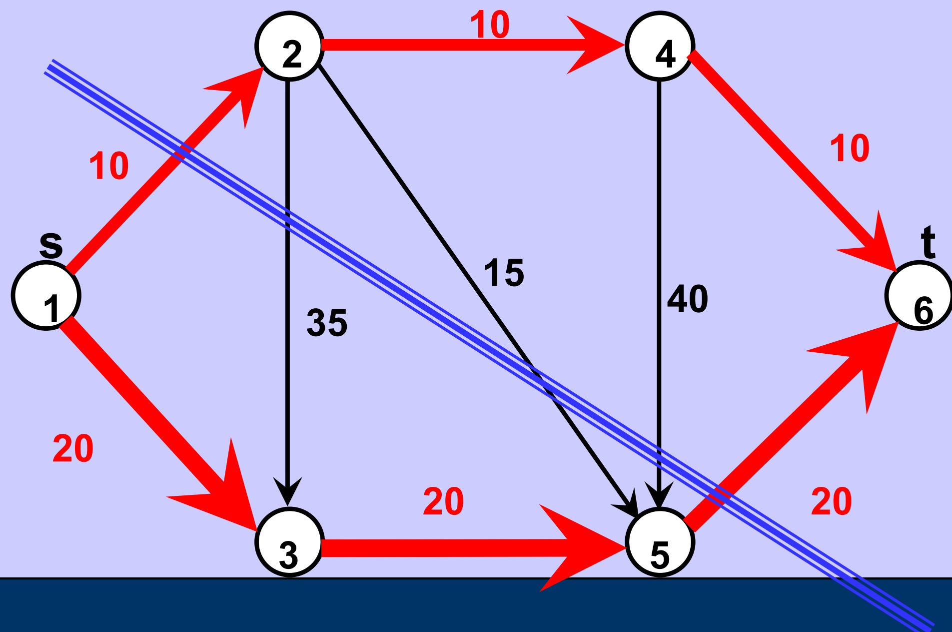


Max-Flow vs. Min-Cut (2)

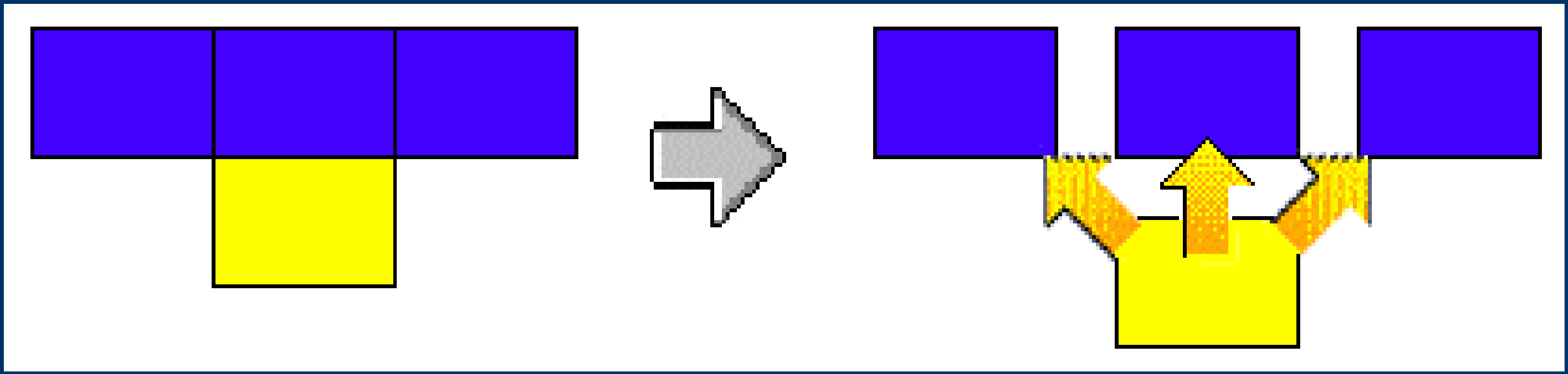
An (s,t)-cut in a network $G = (N, A)$ is a partition of N into two disjoint subsets S and T such that $s \in S$ and $t \in T$.

The capacity of a cut (S,T) is $\sum_{i \in S} \sum_{j \in T} u_{ij}$.

Theorem 4 *The value of a maximum (s,t)-flow is equal to the capacity of a minimum (s,t)-cut.*

Maximum Flow

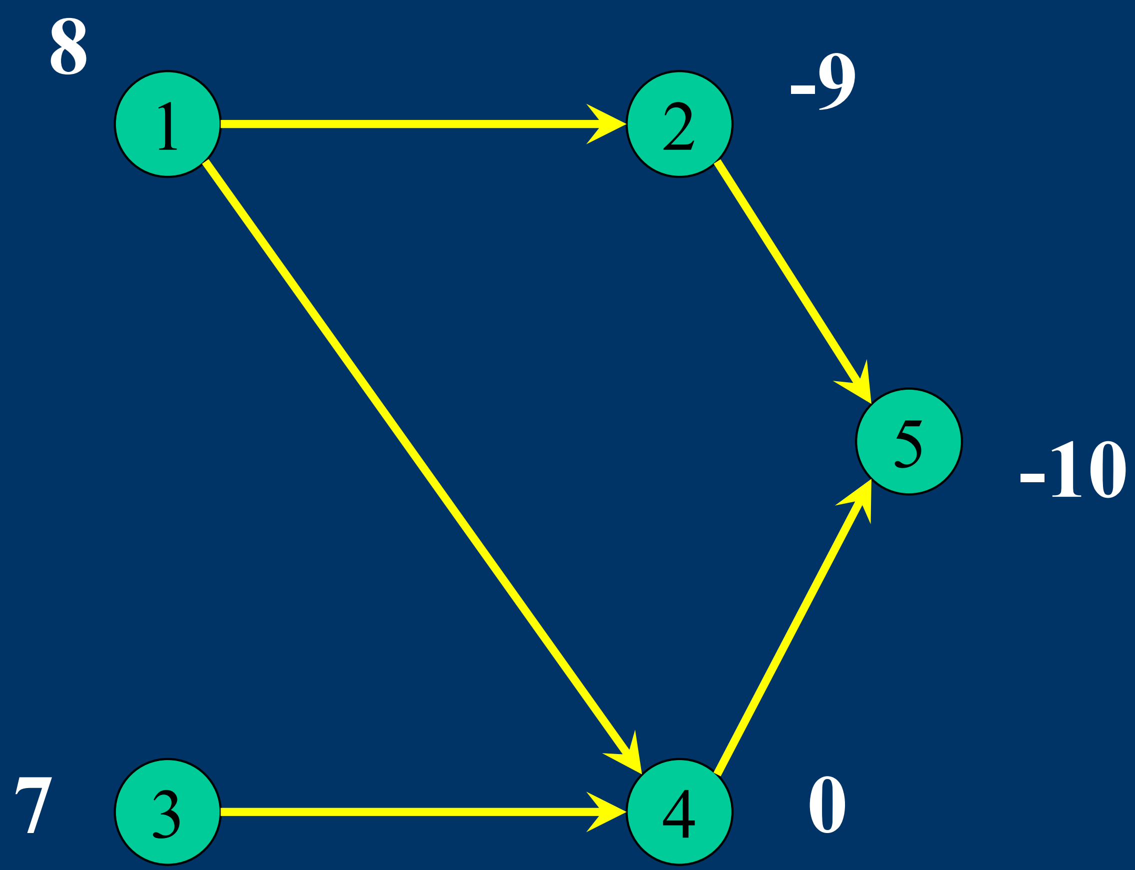
Open Pit Mining



Maximum Flow

Open Pit Mining (2)

Small Example



Maximum Flow

Open Pit Mining (2)

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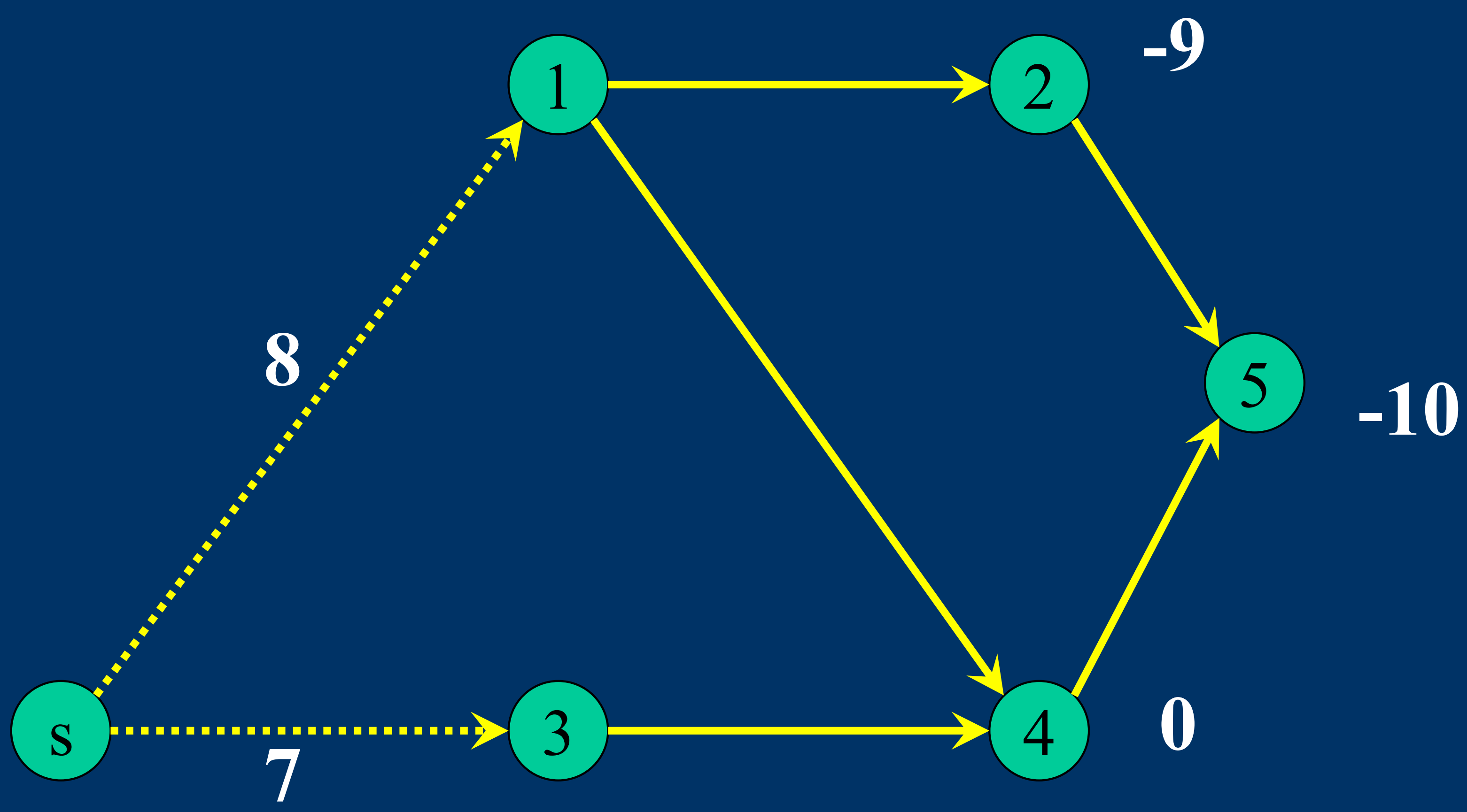

Maximum Flow

Open Pit Mining (2)

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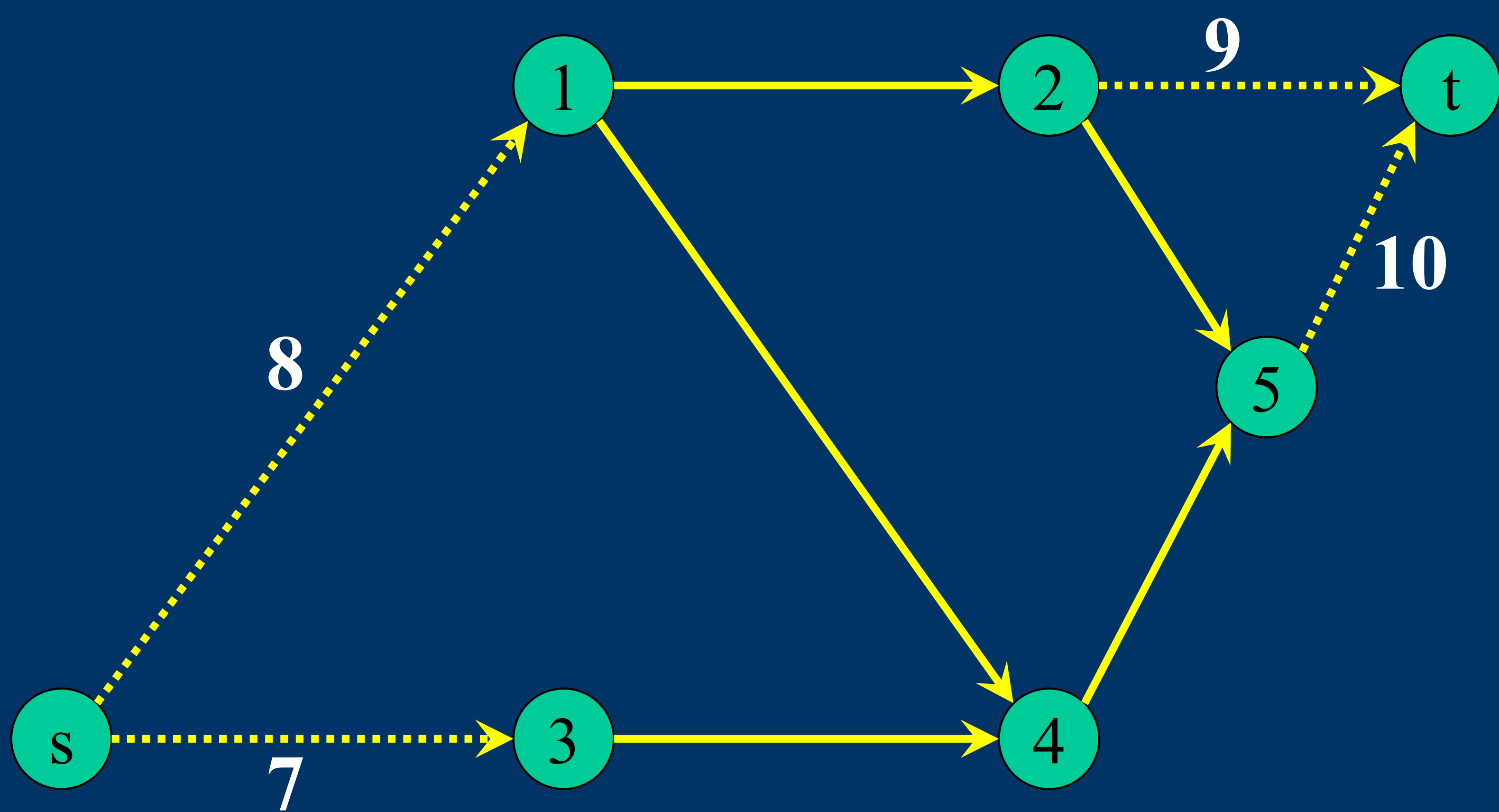

Maximum Flow

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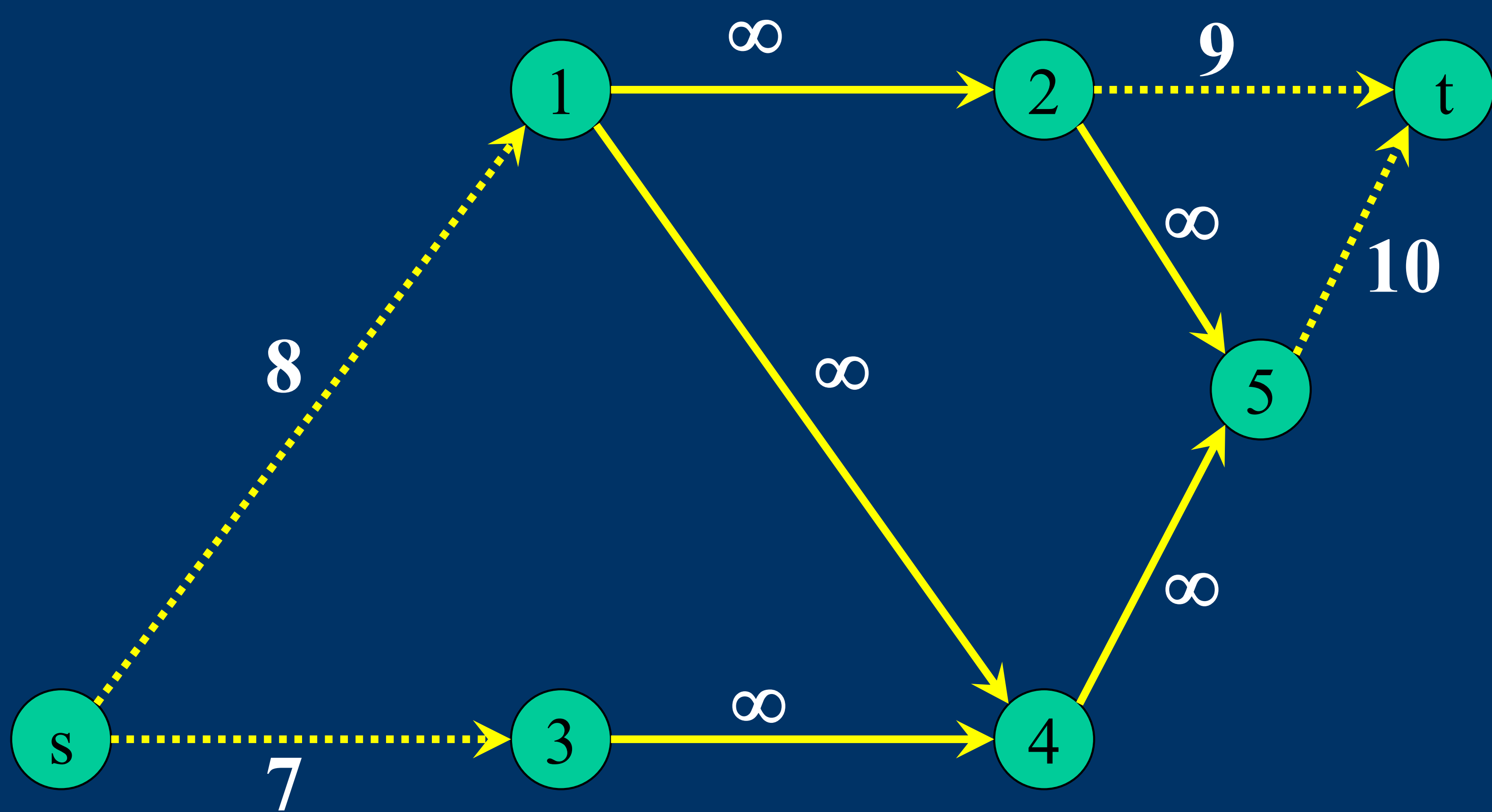


Maximum Flow

Open Pit Mining (3)

- There is a $1:1$-correspondence between feasible sets of blocks and (s,t)-cuts of finite capacity.
- A feasible set B of blocks corresponds to the cut $\{s\} \cup B$.
- The weight of a feasible set B of blocks is $w(B) = \sum_{i \in B^+} w_i - \sum_{i \in B^-} |w_i|$.
- The capacity of the cut $\{s\} \cup B$ is $\sum_{i \in \overline{B}^+} w_i + \sum_{i \in B^-} |w_i|$.
- Hence, $w(B) + \text{cap}(\{s\} \cup B) = \sum_{i : w_i > 0} w_i$.

Min-Cost Flow

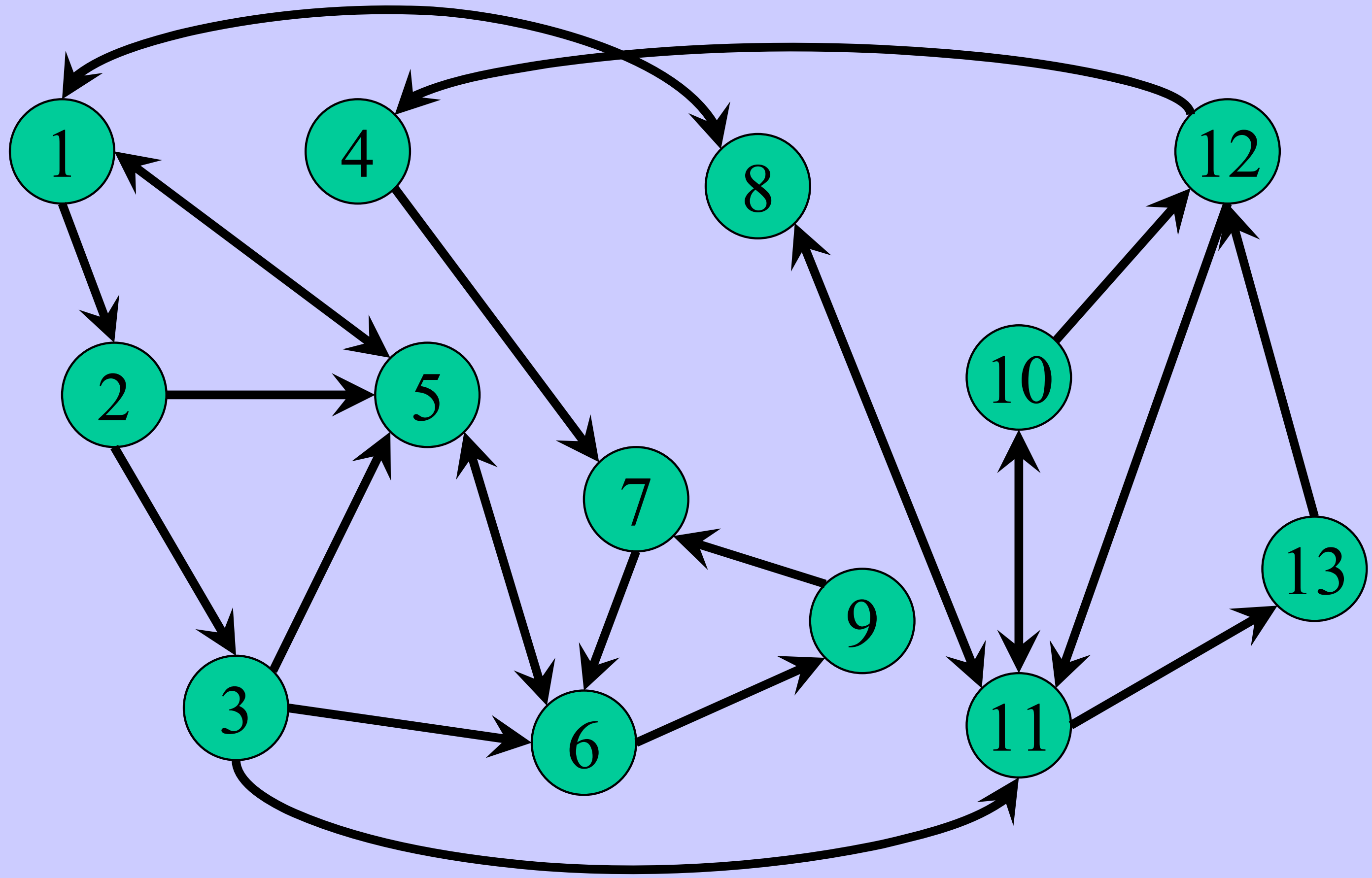
Passenger Routing

- United Airlines has seven daily flights from BOS to SFO, every two hours, starting at 7am.
- Capacities are **100**, **100**, **100**, **150**, **150**, **150**, and ∞.
- Passengers suffering from overbooking are diverted to later flights.
- Delayed passengers get $**200** plus $**20** for every hour of delay.
- Suppose that today the first six flighs have **110**, **160**, **103**, **149**, **175**, and **140** confirmed reservations.

Determine the most economical passenger routing strategy!

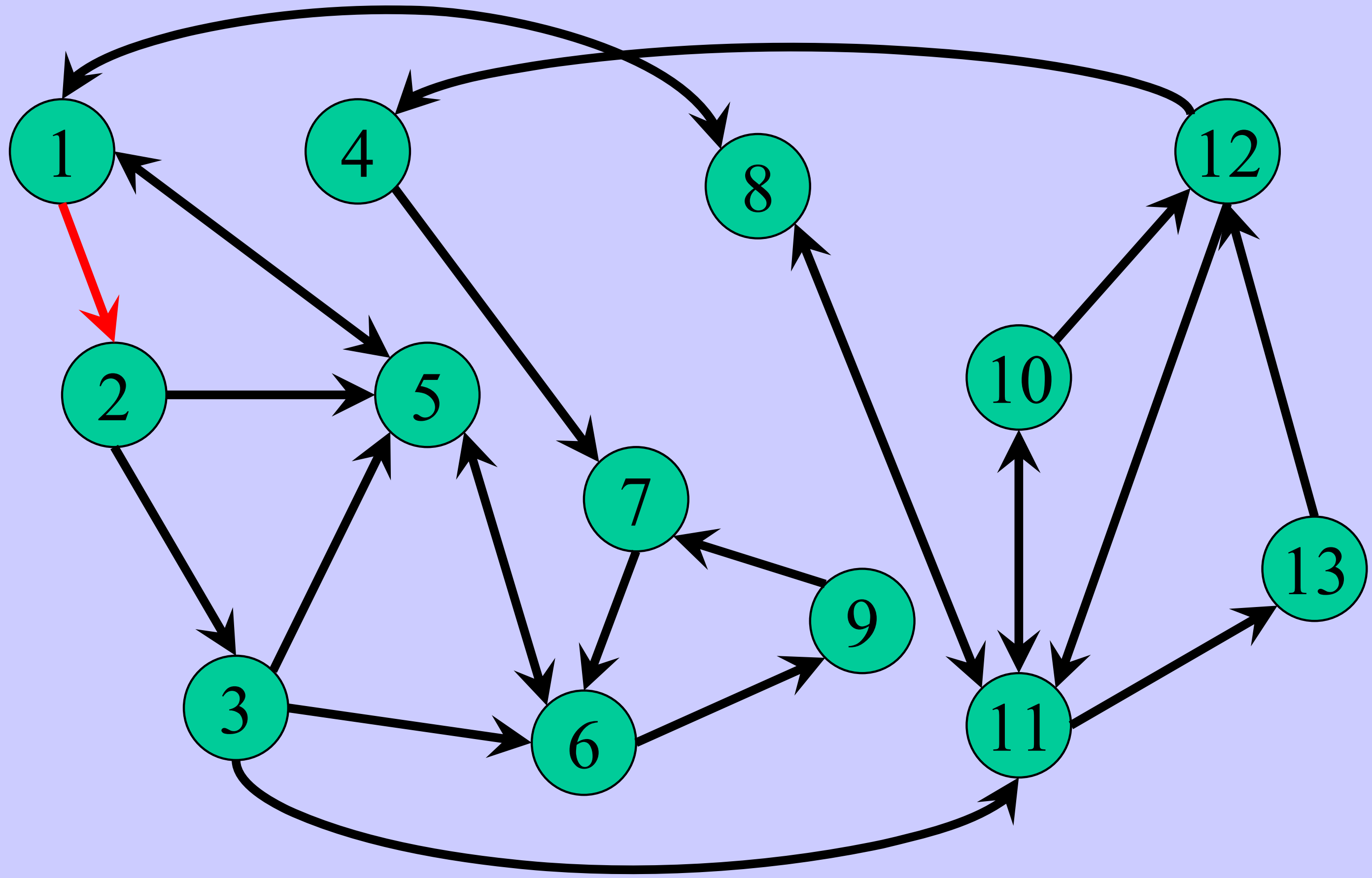
Min-Cost Flow

Postman Problem



Min-Cost Flow

Postman Problem



Min-Cost Flow

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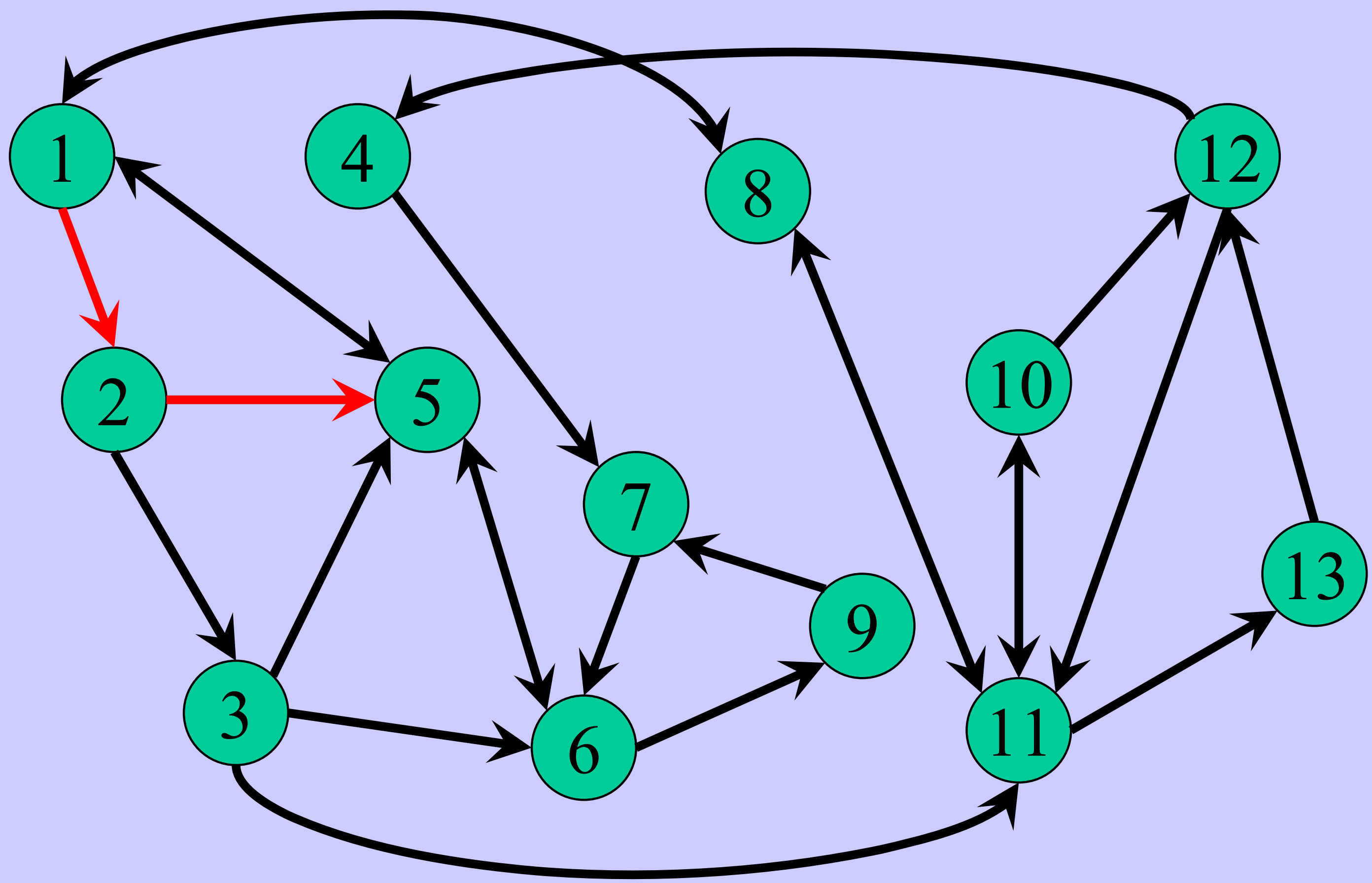

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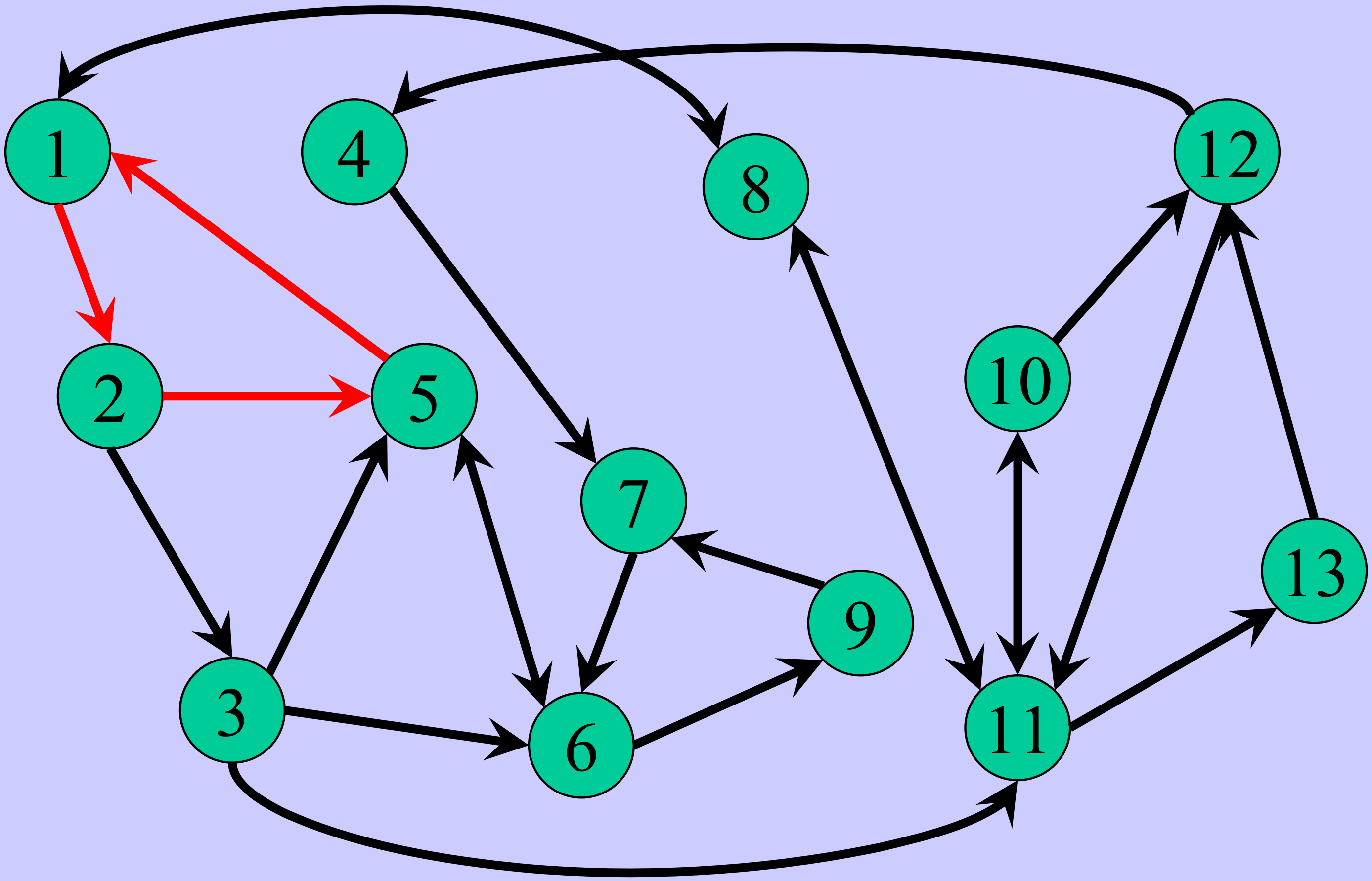

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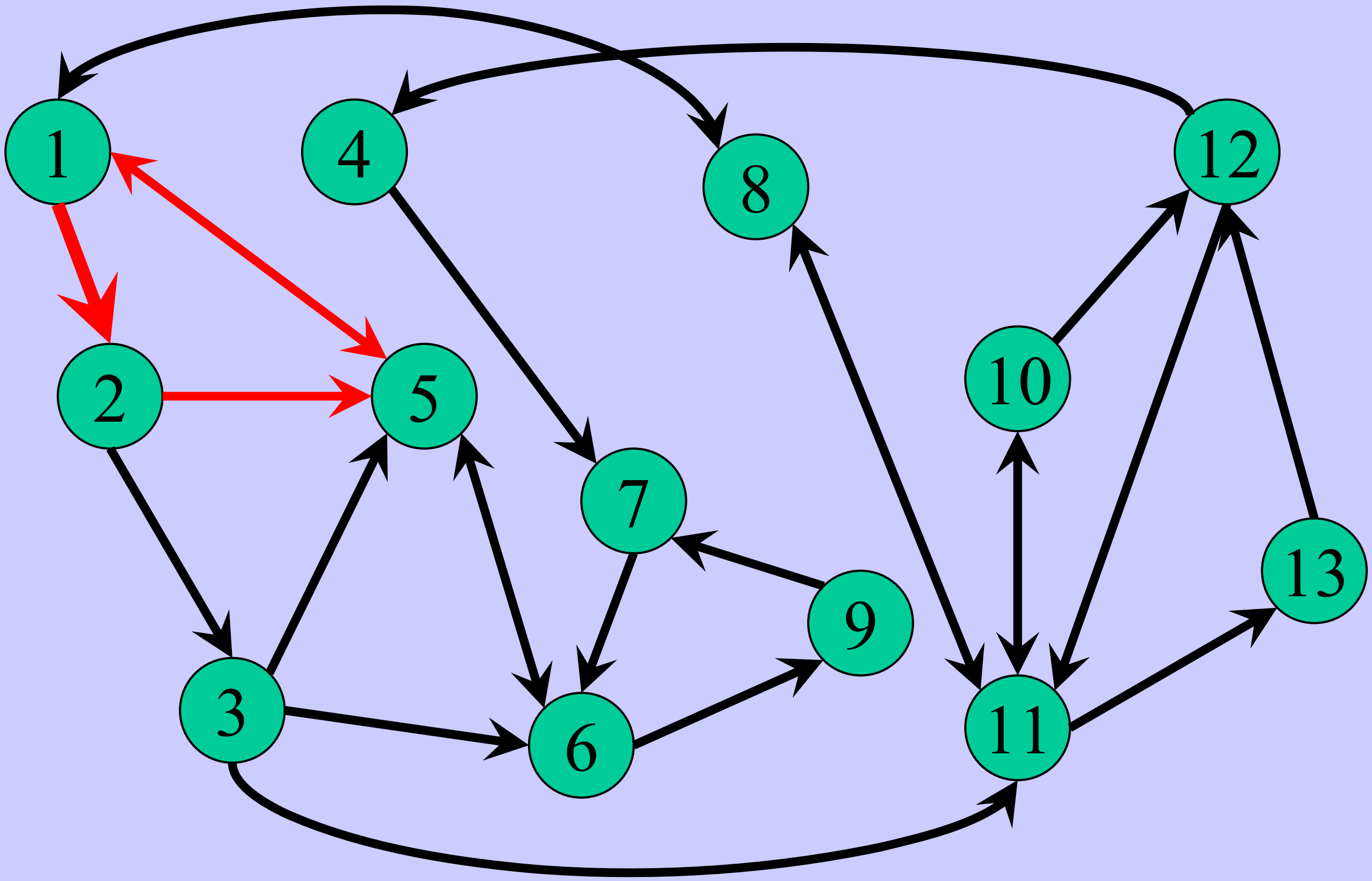

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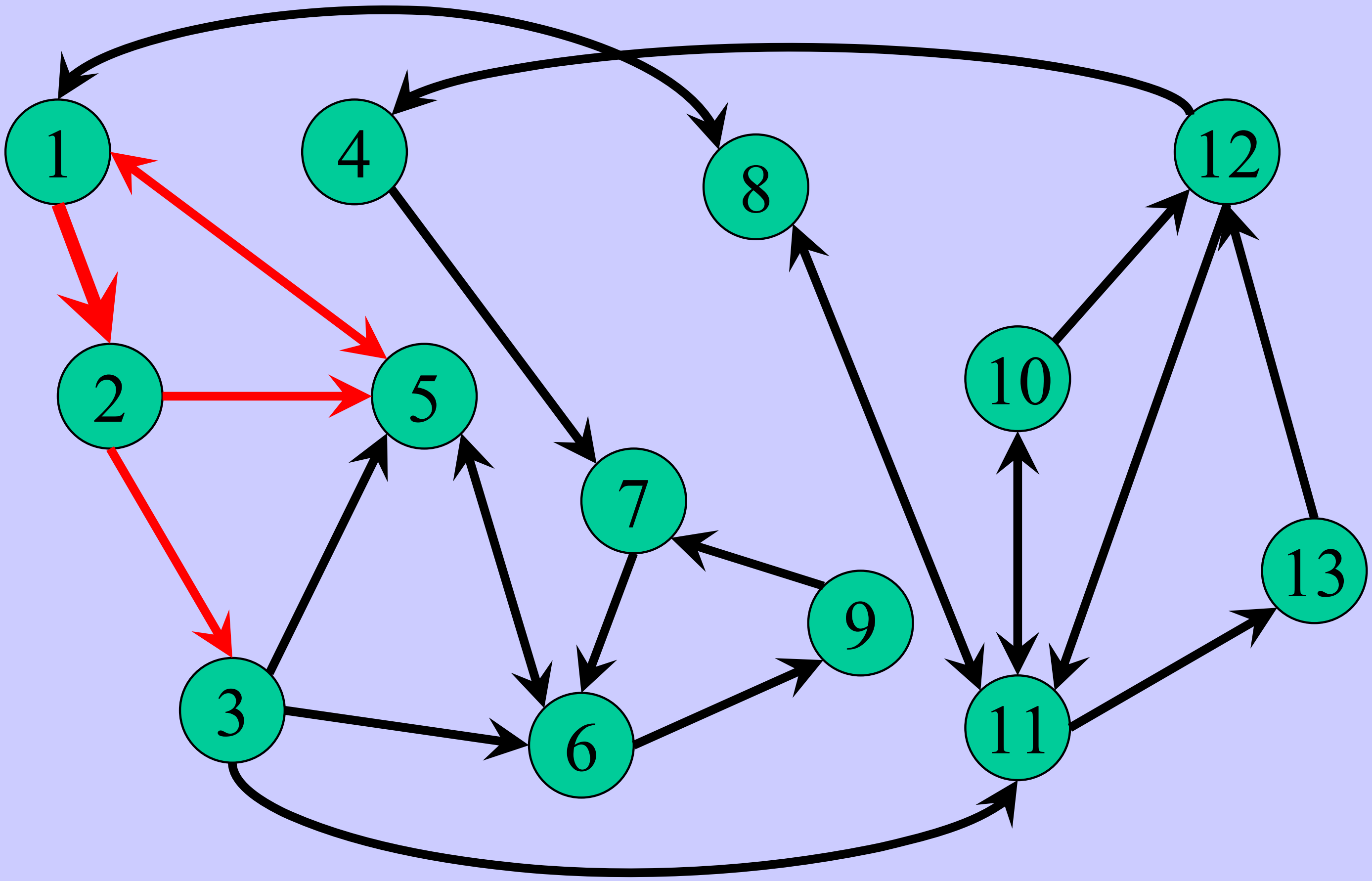

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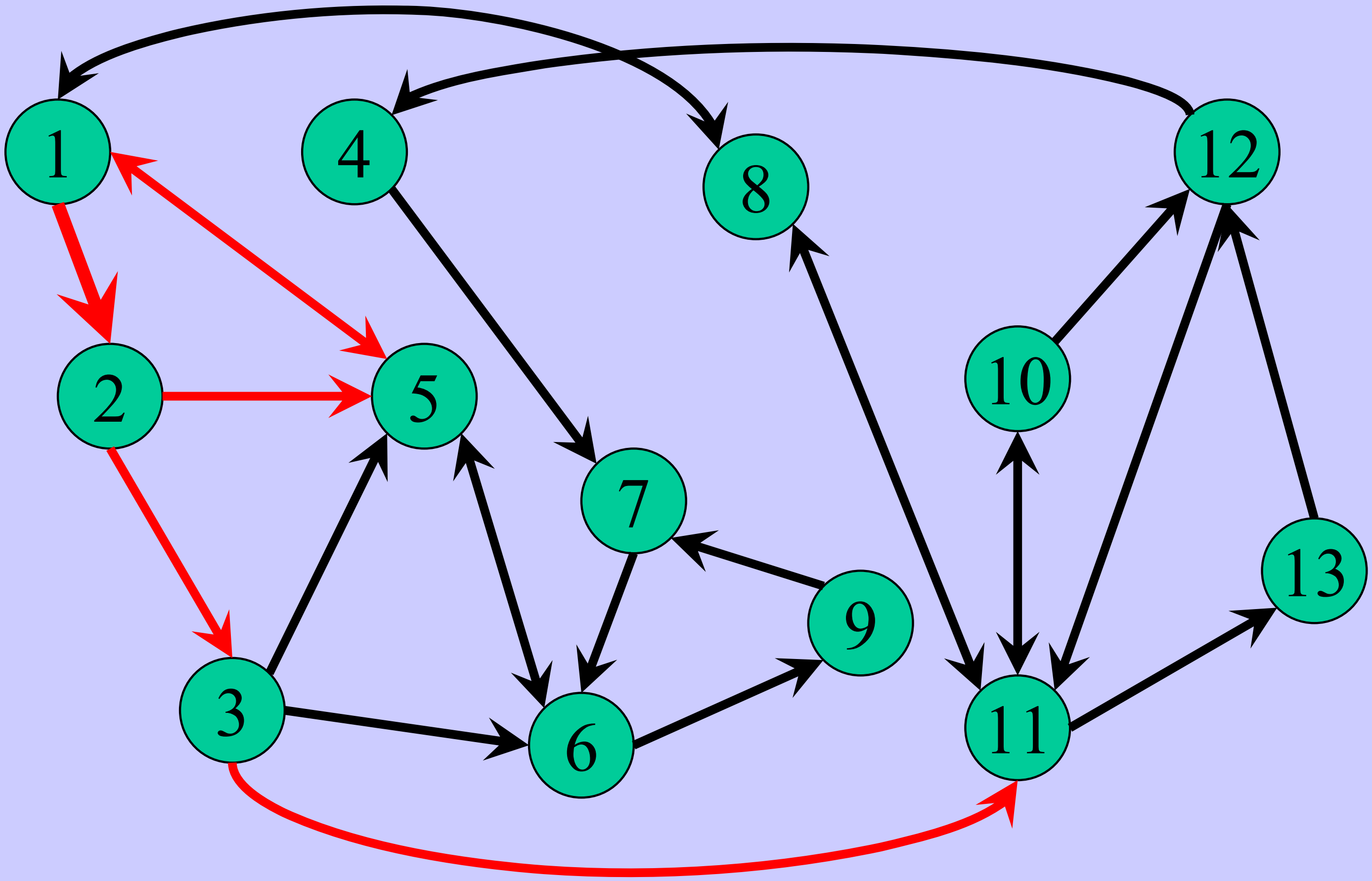

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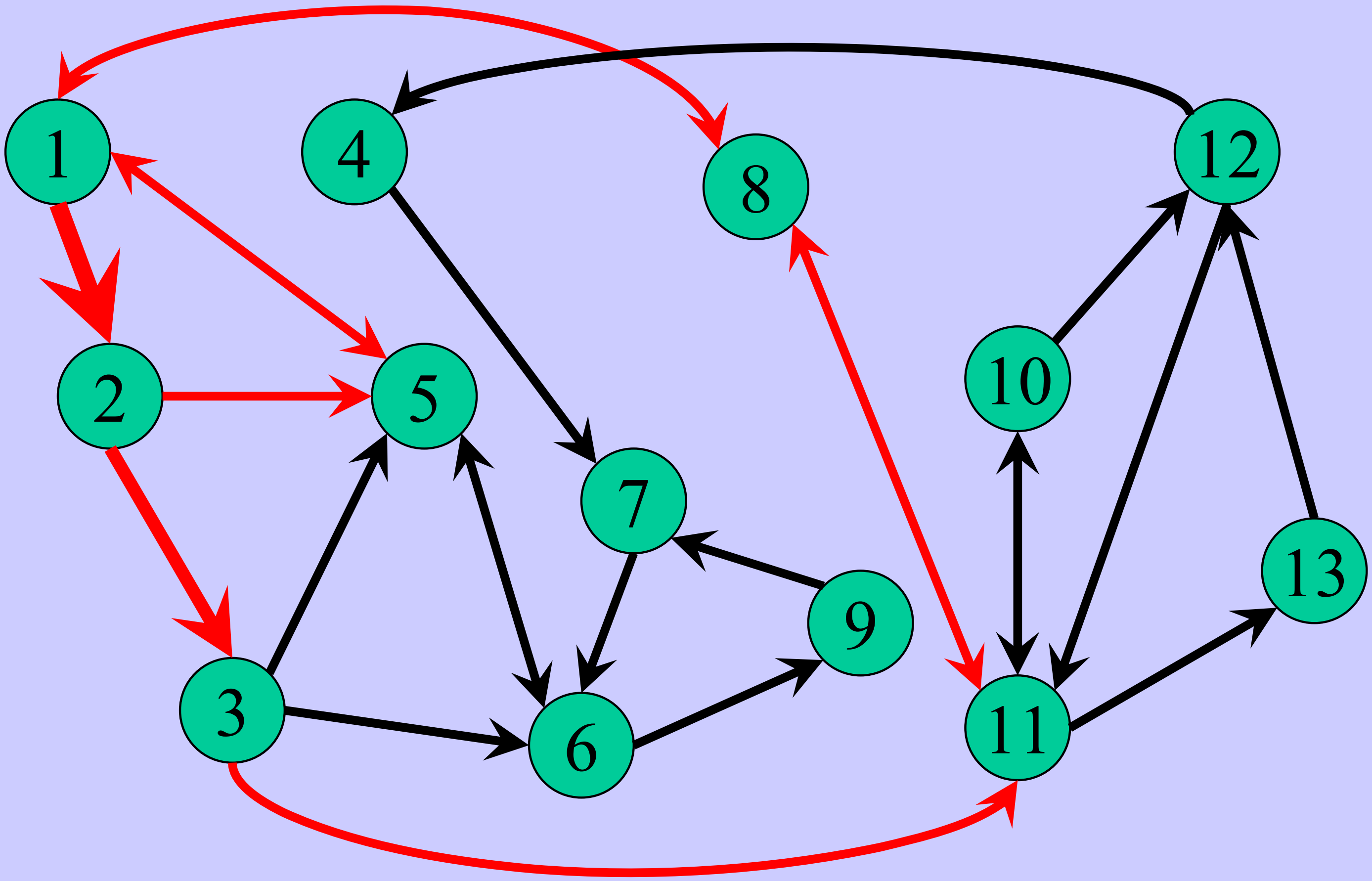

Min-Cost Flow

Postman Problem (2)

- In an optimal walk, a postal carrier might traverse arcs more than once.
- Any carrier walk must satisfy the following conditions:

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for all } i \in N$$

$$x_{ij} \geq 1 \quad \text{for all } (i,j) \in A$$

- Here, x_{ij} is the # of times the carrier traverses arc (i,j).

If we get x, how can we reconstruct a walk?

Further Models

- Minimum spanning tree problems,
- Matching problems,
- Generalized flow problems,
- Multicommodity flow problems,
- Constrained shortest path problems,
- Unsplittable flow problems,
- Network design problems,
- . . .