Problem Set 5

1. (To be solved individually)

   Consider a commodity with the price process P_t following

   $$P_t - \overline{P} = \rho(P_{t-1} - \overline{P}) + \varepsilon_t, \qquad t = 1, 2, ..., T, \qquad |\rho| < 1$$

   where ε_t are IID $\mathcal{N}(0, \sigma_\varepsilon^2)$ shocks.

   You can trade in the risk-free asset with constant interest rate r_f and in a futures contract with futures price Φ_t following

   $$\Phi_t - \overline{\Phi} = \kappa(\Phi_{t-1} - \overline{\Phi}) + u_t, \qquad |\kappa| < 1$$

   where u_t are IID $\mathcal{N}(0, \sigma_u^2)$ shocks, and $\mathrm{corr}(\varepsilon_t, u_t) = \gamma$. You start with initial capital equal to P_0 and your objective is to hedge, as close as possible, the terminal value of the commodity price using the futures contract, i.e., you want to minimize

   $$\mathrm{E}_0\left[(P_T - W_T)^2\right]$$

   (a) Set up this problem as a DP. Be explicit about the dynamic budget constraint on your portfolio and the state space.

   (b) Write down the Bellman equation.

   (c) Compute the value function for $t = T - 1$ and $t = T - 2$.

2. (To be solved in a group) Consider a futures contract with price process under risk-neutral probability

   $$\ln P_t - \ln \overline{P} = \rho(\ln P_{t-1} - \ln \overline{P}) + \varepsilon_t, \qquad t = 1, 2, ..., T, \qquad |\rho| < 1$$

   where ε_t are IID $\mathcal{N}(0, \sigma^2)$ shocks under **Q**. Assume that the interest rate is constant, equal to r_f. Consider an American option on this futures contract, which can be exercised at any time period with the payoff

   $$\max(0, K - P_t)$$

   The option matures at T.

   Assume the following parameter values

   $$T = 52, \; r_f = 0.001, \; \rho = 0.995, \; \overline{P} = \$10, \sigma = 0.03, \; K = \$9$$

   Let the initial futures price be $P_0 = \$10.5$.

(a) Consider a European option with the same payoff and the same parameters. Compute the price of this option numerically on a grid (Hint: use the code for numerical DP, available in the Lecture Notes section of the OCW page, to help you approximate the dynamics of futures prices on a grid).

(b) Check the accuracy of your answer using Monte Carlo simulation. Refine the grid so your answer is accurate up to a penny.

(c) Compute numerically the price of the American option.

(d) At time $t = 26$, under what conditions would it be optimal to exercise the option?

(e) What is the early exercise premium for the American option above, defined as the difference between the price of the American option and the otherwise identical European option?

3. (To be solved individually) (Optional, extra credit) Consider a binomial-tree model of stock returns. Assume that the steps of the tree are given by

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$$

and the probability of the "up" move is

$$p = \frac{e^{\mu\Delta t} - d}{u - d}$$

where

$$\sigma = 0.2, \quad \mu = 0.1, \quad \Delta t = \frac{1}{52}$$

The time step is Δt. (This tree is supposed to model weekly price changes).

Assume that the short-term interest rate is constant, and the single-period simple interest rate is

$$r = \frac{0.04}{52}$$

Consider and investor with initial wealth $W_0 = 1$ maximizing expected utility

$$E_0[\ln W_T]$$

Assume $T = 2$ (the investor has a two-year horizon, so there are 2×52 time periods in this problem). The investor is subject to a constraint that the terminal portfolio value cannot fall below 0.9:

$$W_T \geq 0.9$$

(E.g., a pension fund with a floor on portfolio value dictated by liabilities). Your objective is to find the optimal dynamic portfolio strategy for this investor using the static approach.

(a) Formulate the static optimization problem for this investor.

(b) Relax the budget constraint using the Lagrange multiplier λ and derive the optimal terminal wealth $W_T^\star$ as a function of the SPD and λ. Make sure you satisfy the lower bound $W_T \geq 0.9$: the first-order optimality conditions hold only if the inequality constraint is not binding. Comment on the relationship between $W_T^\star$ and the SPD π_T: what's the intuition?

(c) Find the value of λ such that $W_T^\star$ satisfies the budget constraint. For that you need to compute the time-0 market value of the state-contingent cash flow $W_T^\star$ and make sure it equals W_0. You can use backward induction on the tree or Monte Carlo simulation. To use backward induction, show explicitly how the optimal choice $W_T^\star$ depends on the terminal stock price (and not on the entire path of the stock price between 0 and T).

(d) Having derived the optimal terminal wealth, you can now compute the optimal portfolio policy. Compute the optimal stock holding at time 0. For that you need to compute the optimal wealth at nodes u and d at time Δt and use the formula for option replication on a binomial tree. Again, you can use backward induction on the tree or Monte Carlo simulation. Note that Monte Carlo simulation may require many trajectories to achieve high degree of accuracy.

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