Key Points: Derivatives

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Discrete Models

- Definitions of SPD (π) and risk-neutral probability (**Q**).
- Absence of arbitrage is equivalent to existence of the SPD or a risk-neutral probability:

$$P_t = \mathrm{E}_t^{\mathbf{P}}\left[\sum_{u=t+1}^{T} \frac{\pi_u}{\pi_t} D_u\right] = \mathrm{E}_t^{\mathbf{Q}}\left[\sum_{u=t+1}^{T} \frac{B_t}{B_u} D_u\right]$$

- Price of risk: under Gaussian **P** and **Q** distributions,

$$\varepsilon_t^{\mathbf{Q}} = \varepsilon_t^{\mathbf{P}} + \eta_t$$

- Log-normal model (discrete version of Black-Scholes):

$$\mu_t - r_t = \sigma_t \eta_t$$

Stochastic Calculus

- Brownian motion, basic properties (IID Gaussian increments, continuous trajectories, nowhere differentiable).
- Quadratic variation. $[Z]_T = T$. Heuristically,

$$(dZ_t)^2 = dt.$$

- Stochastic integral.
- Ito's lemma:

$$df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t}\, dt + \frac{\partial f(t, X_t)}{\partial X_t}\, dX_t + \frac{1}{2}\frac{\partial^2 f(t, X_t)}{\partial X_t^2}\, (dX_t)^2$$

- Multivariate Ito's lemma.

$$df(t, X_t, Y_t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial Y_t}\, dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, (dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial Y_t^2}\, (dY_t)^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t}\, dX_t\, dY_t$$

Black-Scholes Model

- Arbitrage-free pricing of options by replication.
- European option with payoff $H(S_T)$.
- Replicating portfolio delta is

$$\theta_t = \frac{\partial f(t, S_t)}{\partial S_t}$$

$$-r\, f(t, S) + \frac{\partial f(t, S)}{\partial t} + rS \frac{\partial f(t, S)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f(t, S)}{\partial S^2} = 0$$

with the boundary condition $f(T, S) = H(S)$.

Pricing by Replication: Limitations

- In many models cannot derive a unique price for a derivative.
- Term structure models, stochastic volatility.
- Price assets relative to each other. Replication argument combined with assumptions on prices of risk.
- Alternatively, specify dynamics directly under **Q**.

Risk-Neutral Pricing

- General pricing formula

$$P_t = \mathrm{E}_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r_s\, ds\right) H_T\right]$$

- Need to specify dynamics of the underlying under **Q**.
- If underlying is a stock, only one way to do this: set expected return to r.
- **Q** dynamics is related to **P** through price of risk

$$dZ_t^{\mathbf{P}} = -\eta_t\, dt + dZ_t^{\mathbf{Q}}$$

- Risk premium

$$\mathrm{E}_t^{\mathbf{P}}\left[\frac{dS_t}{S_t}\right] - r_t\, dt = \mathrm{E}_t^{\mathbf{P}}\left[\frac{dS_t}{S_t}\right] - \mathrm{E}_t^{\mathbf{Q}}\left[\frac{dS_t}{S_t}\right]$$

Risk-Neutral Pricing and PDEs

- Derive a PDE on derivative prices using Ito's lemma.
- One-factor term structure model

$$\mathrm{E}_t[df(t, r_t)] = r_t f(t, r_t)\, dt$$

- Vasicek model:

$$dr_t = -\kappa(r_t - \bar{r})\, dt + \sigma\, dZ_t^{\mathbf{Q}}$$

- $f(t, r_t)$ must satisfy the PDE

$$\frac{\partial f(t,r)}{\partial t} - \kappa(r - \bar{r})\frac{\partial f(t,r)}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(t,r)}{\partial r^2} = rf(t,r)$$

  with the boundary condition

$$f(T, r) = 1$$

- Expected bond returns satisfy

$$\mathrm{E}_t\left(\frac{dP(t,T)}{P(t,T)}\right) = (r_t + \sigma_t^P \eta_t)\, dt$$

Monte Carlo Simulation

- Random number generation: inverse transform, acceptance-rejection method.
- Variance reduction: antithetic variates, control variates.
- Intuition behind control variates: carve out the part of the estimated moment that is known in closed form, no need to estimate that by Monte Carlo.
- Good control variates: highly correlated with the variable of interest, expectation known in closed form.
- Examples of control variates: stock price, payoff of similar option, etc.

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