<u>Newsvendor Inventory Problem</u>

Consider a newspaper vendor selling papers on the corner. Each morning, the vendor must decide how many papers to buy at the wholesale price. The vendor then sells the papers during the day at a retail price higher than the wholesale price. At the end of the day, any unused papers can no longer be sold and are scrapped (or possibly sold back to the wholesaler at a price less than the wholesale price). How many papers should the vendor buy?

Characteristics of the problem
   1. Single period problem – ordering decision made every period
   2. Demand is uncertain
   3. Order is placed before demand materializes
   4. We can only order once per period
   5. There is a cost for ordering too much and a cost for ordering too few items.

   Examples
   1.   Fashion products
   2.   Perishable products
   3.   …

Model Basics

Inputs
   • <u>Wholesale cost</u> – the per unit cost for the vendor to purchase the item, denoted by **w**
   • <u>Retail price</u> – the per unit selling price, denoted by **p**
   • <u>Salvage value</u> – the per-unit amount the vendor can receive if a unit cannot be sold; for instance , at the end of the period, an item might be sold back to the wholesaler at a discount, or sold on a secondary market; the salvage price is denoted by **s**.
   • <u>Demand distribution</u> – Let F(x) denote the cumulative distribution function of demand. That is F(x) is the probability that the demand will be less than or equal to x.

Model Intuition

Suppose that we have decided to order Q units. Consider the question: should we order another unit? That is, is it economical to order the Q+1st unit?

We define the incremental profit from the Q+1st unit as follows:

$$\Delta(Q) = E\left[profit(Q+1)\right] - E\left[profit(Q)\right].$$

It is economical to order the Q+1st unit if and only if $\Delta(Q) \geq 0.$

For this analysis, we assume that the Q+1st unit is reserved to serve the Q+1st demand.

$\Delta(Q)$ has three components:

- A cost of c, namely the cost to purchase the Q+1st unit.
- A revenue of p, if the Q+1st unit is sold.
- A salvage value of s, if the Q+1st unit is not sold.

Thus in expectation, we can write:

$$\Delta(Q) = -c + p \times \Pr\left[sell\ Q+1^{st}\ unit\right] + s \times \Pr\left[do\ not\ sell\ Q+1^{st}\ unit\right]$$

$$= -c + p \times \Pr\left[sell\ Q+1^{st}\ unit\right] + s \times \left(1 - \Pr\left[sell\ Q+1^{st}\ unit\right]\right)$$

$$= (p-c) \times \Pr\left[sell\ Q+1^{st}\ unit\right] - (c-s) \times \left(1 - \Pr\left[sell\ Q+1^{st}\ unit\right]\right)$$

Now we find:

$$\Delta(Q) \geq 0 \Rightarrow \Pr\left[sell\ Q+1^{st}\ unit\right] \geq \frac{(c-s)}{(p-c)+(c-s)}$$

The implication for this analysis is that we should increase Q until this condition does not hold; thus, we choose the first Q such that

$$\Pr\left[sell\ Q+1^{st}\ unit\right] < \frac{(c-s)}{(p-c)+(c-s)}.$$

Now what is $\Pr\left[sell\ Q+1^{st}\ unit\right]$? This is the probability that demand is at least Q+1 units. The cumulative distribution function is defined as $F(x) = \Pr\left[demand \leq x\right].$ Hence, we have

$$\Pr\left[sell\ Q+1^{st}\ unit\right]=1-\Pr\left[demand \le Q\right]=1-F(Q).$$

Alternatively, if we have a discrete probability distribution, we can write:

$$\Pr\left[sell\ Q+1^{st}\ unit\right]=\sum_{j=Q+1}^{\infty}\Pr\left[demand = j\right]=1-\sum_{j=0}^{Q}\Pr\left[demand = j\right]$$

Alternative Interpretation

- We often term **c - s** to be the <u>overage cost</u> – the incremental per-unit cost for any items that cannot be sold.
- We then term **p - c** to be the <u>underage cost</u>– the incremental per-unit cost for not meeting demand.

The above result can then be expressed, more generally, as:

We choose the first Q such that

$$\Pr\left[sell\ Q+1^{st}\ unit\right]<\frac{overage\_cost}{underage\_cost+overage\_cost}.$$

Sometimes this result will be expressed equivalently as:

We choose the first Q such that

$$\Pr\left[do\ not\ sell\ Q+1^{st}\ unit\right]=F(Q)\ge\frac{underage\_cost}{underage\_cost+overage\_cost}.$$

Useful Result

If demand is normally distributed, then we can find Q in one of two ways:

- Directly from Excel:
$$Q = \text{NORMINV}\left(\mu,\sigma,\frac{\text{Underage Cost}}{\text{Underage Cost + Overage Cost}}\right)$$
- Using Excel to find z* first:

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$$z^* = \text{NORMSINV}\left(\frac{\text{Underage Cost}}{\text{Underage Cost + Overage Cost}}\right)$$

and then letting $Q = \mu + (z^*)(\sigma)$

Discrete Demand Example – Theodore's Gift Shop

Problem Statement

Theodore's gift shop places orders for Christmas items during a trade show in July. One item to be ordered is a dated sterling silver tree ornament. The ornament will sell for $80. The best estimate for demand is:

Demand	Probability
5	0.20
6	0.25
7	0.30
8	0.25

The ornaments cost $55 when ordered in July. Ornaments unsold by Christmas are marked down to half price and always sell during January. How many ornaments should be ordered?

Calculations

- Overage cost = $55 per ornament- $40 per ornament = $15 per ornament

- Underage cost = $80 per ornament - $55 per ornament = $25 per ornament

- We choose the first Q such that

$$\Pr\left[sell\ Q+1^{st}\ unit\right] < \frac{overage\_cost}{underage\_cost + overage\_cost} = \frac{\$15}{\$25+\$15} = 0.375$$

- Table of cumulative probability distribution

Demand	Cumulative Probability=F(x)	1-F(x)
5	0.20	0.80
6	0.45	0.55
7	0.75	0.25
8	1.00	0

- To find Q, we compare:

$$\Pr\left[sell\ Q+1^{st}\ unit\right]=1-\Pr\left[demand\le Q\right]=1-F(Q).$$

To the critical fractile 0.375

In this case, Q = 7 ornaments. (as the probability that we will see the 8th unit is only 0.25)

- With Q = 7 ornaments, what is the total expected profit?

Demand	Probability	Revenue	Expected Revenue
5	0.20	5(80)+2(40)=480	(0.20)(480)=96
6	0.25	6(80)+1(40)=520	(0.25)(520)=130
7	0.30	7(80)+0(40)=560	(0.30)(560)=168
8	0.25	7(80)+0(40)=560	(0.25)(560)=140

Total Expected Revenue = 534

Total expected profit = total expected revenue – cost = 534 – 7(55) = 149

- As a point of reference:

For Q = 6, expected revenue = 472, cost = 330 and expected profit = 142

For Q = 8, expected revenue = 584, cost = 440 and expected profit = 144

Continuous Demand Example – Johnson Shoe Company

Problem Statement

• The Johnson Shoe Company buys shoes for $40 per pair and sells them for $60 per pair. If there are surplus shoes left at the end of the season, all shoes are expected to be sold at the sale price of $30 per pair. How many shoes should the Johnson Shoe Company buy?

Calculations

• Overage cost = $40 per pair - $30 per pair = $10 per pair

• Underage cost = $60 per pair - $40 per pair = $20 per pair

• We choose the first Q such that

$$\Pr\left[sell\ Q+1^{st}\ unit\right] < \frac{overage\_cost}{underage\_cost+overage\_cost} = \frac{\$10}{\$20+\$10} = 0.333$$

• Suppose demand is normally distributed with a mean of 500 units and a standard deviation of 100 units/season. We then want to find the first value of Q such that $1 - F(Q) < 0.33$. That is, we need to find the z-score such that 1/3 of the area under the curve is beyond z.

Equivalently we find z such that $\Phi(z) = 0.67$ where $\Phi(\ )$ is the cumulative distribution function for a standard normal distribution.

Looking up $\Phi(z) = 0.67$ F(z) yields a z of 0.44.

Q = mean + (0.44)*(standard deviation) = 500+ (.44)(100) = 544 units

Multi-item Style Goods Problem with Capacity Constraint

- Suppose that instead of buying a single item, you had to place orders for several items with the condition that the total order quantity across all items can not exceed your capacity C.

Example

- Consider a newsstand that sells three scholarly journals: *Management Science* (MS), *Operations Research* (OR), and *Manufacturing & Service Operations Management* (M&SOM). For simplicity, let's assume that the journals have the same costs and revenues. Journals cost $1.00 to buy and they are sold at a retail price of $4.00. Journals left unsold at the end of the season can be returned to the publisher for $0.50.

- This implies the underage cost is $3.00 ($4.00 - $1.00) and the overage cost is $0.50 ($1.00 - $0.50). Therefore, the critical fractile is 0.14.

- We assume normally distributed demand with parameters shown in table; then we can calculate an order quantity for each, as shown in Table

Journal	Mean	Sigma	CF	z-score	Q*
MS	80	40	0.14	1.07	123
OR	50	30	0.14	1.07	82
MSOM	20	15	0.14	1.07	36

Total   241

- Now let's assume that there is a capacity limit of 200 total magazines. How would you determine how many to order?

  Most people would guess a linear allocation based on the journal's percentage composition of the total optimal order size. That is,

$$\text{Linear Order Size} = \left(\frac{Q*}{241} \right) (200)$$

Journal	Mean	Sigma	Q*	% of Total	Linear Order Size
MS	80	40	123	51%	102
OR	50	30	82	34%	68
MSOM	20	15	36	15%	30

Total   241   200

- While the linear allocation approach is intuitively appealing, it is incorrect because the resulting order quantities now have different z-scores. This implies that each of the journals now has a different probability of fully satisfying the demand (termed ***fill rate*** here). Recall that before the

capacity constraint was imposed, each journal's order quantity was set to ensure a 86% fill rate. The following table shows the new fill rates and z-score values for each journal:

Journal	Mean	Sigma	Q*	% of Total	Linear (L) Order Size	L's z-score	L's Fill Rate
MS	80	40	123	51%	102	.55	71%
OR	50	30	82	34%	68	.60	73%
MSOM	20	15	36	15%	30	055	75%
			Total 241		200		

- The correct approach is to find the single z-value that equalizes the fill rate for all three products while not exceeding the capacity constraint. This can be written as follows:

Find the z such that $\sum_i (\mu_i + z\sigma_i) = C$ where i is an index denoting the products. In our example, this can be written out as:

$$80 + 40z + 50 + 30z + 20 + 15z = 200$$
$$85z = 50$$
$$z = 0.59$$

This yields the following ordering quantities

Journal	Mean	Sigma	Order Amount
MS	80	40	103.5
OR	50	30	67.6
MSOM	20	15	28.8
		Total	200
		z-score	0.59

You can find the z-score by using the Goal Seek function of excel. That is, make the Order Amount cells equal to $\mu_i + z\sigma_i$ and the Total cell equal to the sum of the three order amounts. Then set Total equal to 200 by changing the z-score cell.

- If the products do not have the same costs and retail prices, then the problem is much harder. You will want to develop some decision rules to determine what exactly to do.

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