CHAPTER 9 POLAR COORDINATES AND COMPLEX NUMBERS

9.1 Polar Coordinates (page 350)

Circles around the origin are so important that they have their own coordinate system – *polar coordinates*. The center at the origin is sometimes called the "pole." A circle has an equation like $r = 3$. Each point on that circle has two coordinates, say $r = 3$ and $\theta = \frac{\pi}{2}$. This angle locates the point $90°$ around from the x axis, so it is on the y axis at distance 3.

The connection to x and y is by the equations $x = r\cos\theta$ and $y = r\sin\theta$. Substituting $r = 3$ and $\theta = \frac{\pi}{2}$ as in our example, the point has $x = 3\cos\frac{\pi}{2} = 0$ and $y = 3\sin\frac{\pi}{2} = 3$. The polar coordinates are $(r, \theta) = (3, \frac{\pi}{2})$ and the rectangular coordinates are $(x, y) = (0, 3)$.

1. Find polar coordinates for these points – first with $r \geq 0$ and $0 \leq \theta < 2\pi$, then three other pairs (r, θ) that give the same point:

   (a) $(x, y) = (\sqrt{3}, 1)$ (b) $(x, y) = (-1, 1)$ (c) $(x, y) = (-3, -4)$

   - (a) $r^2 = x^2 + y^2 = 4$ yields $r = 2$ and $\frac{y}{x} = \frac{1}{\sqrt{3}} = \tan\theta$ leads to $\theta = \frac{\pi}{6}$. The polar coordinates are $(2, \frac{\pi}{6})$. Other representations of the same point are $(2, \frac{\pi}{6} + 2\pi)$ and $(2, \frac{\pi}{6} - 2\pi)$. Allowing $r < 0$ we have $(-2, -\frac{5\pi}{6})$ and $(-2, \frac{7\pi}{6})$. There are an infinite number of possibilities.
   - (b) $r^2 = x^2 + y^2$ yields $r = \sqrt{2}$ and $\frac{y}{x} = \frac{1}{-1} = \tan\theta$. Normally the arctan function gives $\tan^{-1}(-1) = -\frac{\pi}{4}$. But that is a fourth quadrant angle, while the point $(-1, 1)$ is in the second quadrant. The choice $\theta = \frac{3\pi}{4}$ gives the "standard" polar coordinates $(\sqrt{2}, \frac{3\pi}{4})$. Other representations are $(\sqrt{2}, \frac{11\pi}{4})$ and $(\sqrt{2}, -\frac{5\pi}{4})$. Allowing negative r we have $(-\sqrt{2}, -\frac{\pi}{4})$ and $(-\sqrt{2}, \frac{7\pi}{4})$.
   - (c) The point $(-3, -4)$ is in the third quadrant with $r = \sqrt{9 + 16} = 5$. Choose $\theta = \pi + \tan^{-1}(\frac{-4}{-3}) \approx \pi + 0.9 \approx 4$ radians. Other representations of this point are $(5, 2\pi + 4)$ and $(5, 4\pi + 4)$, and $(-5, 0.9)$.

2. Convert $(r, \theta) = (6, -\frac{\pi}{2})$ to rectangular coordinates by $x = r\cos\theta$ and $y = r\sin\theta$.

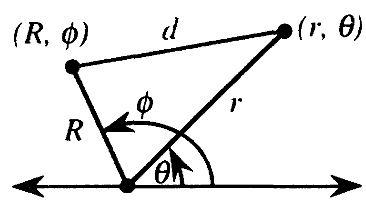
   - The x coordinate is $6\cos(-\frac{\pi}{2}) = 0$. The y coordinate is $6\sin(-\frac{\pi}{2}) = -6$.

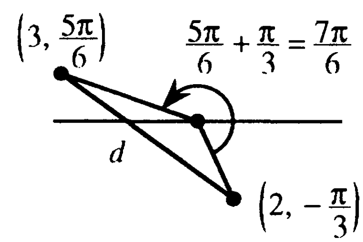
3. The Law of Cosines in trigonometry states that $c^2 = a^2 + b^2 - 2ab\cos C$. Here a, b and c are the side lengths of the triangle and C is the angle opposite side c. Use the Law of Cosines to find the distance between the points with polar coordinates (r, θ) and (R, φ).

   Does it ever happen that c^2 is larger than $a^2 + b^2$?

   - In the figure, the desired distance is labeled d. The other sides of the triangle have lengths R and r. The angle opposite d is $(\varphi - \theta)$. The Law of Cosines gives $d = \sqrt{R^2 + r^2 - 2Rr\cos(\varphi - \theta)}$.

   Yes, c^2 is larger than $a^2 + b^2$ when the angle $C = \varphi - \theta$ is larger than $90°$. Its cosine is negative. The next problem is an example. When the angle C is *acute* (smaller than $90°$) then the term $-2ab\cos C$ reduces c^2 below $a^2 + b^2$.



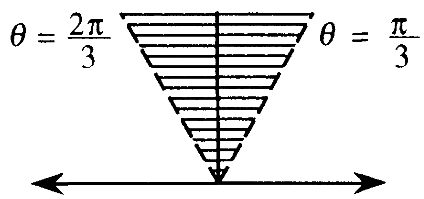


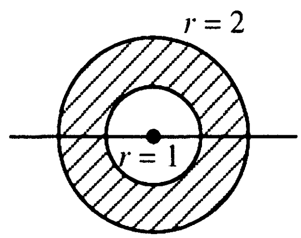
3′. Use the formula in Problem 3 to find the distance between the polar points $(3, \frac{5\pi}{6})$ and $(2, -\frac{\pi}{3})$.

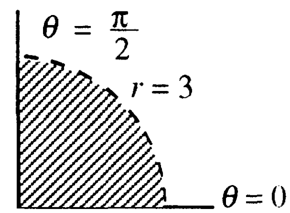
- $d = \sqrt{3^2 + 2^2 - 2 \cdot 3 \cdot 2\cos(\frac{5\pi}{6} - (-\frac{\pi}{3}))} = \sqrt{13 - 12\cos\frac{7\pi}{6}} = \sqrt{13 + 6\sqrt{3}} \approx 4.8.$

4. Sketch the regions that are described in polar coordinates by
   (a) $r \geq 0$ and $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ (b) $1 \leq r \leq 2$ (c) $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq r < 3$.

   - The three regions are drawn. For (a), the dotted lines mean that $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are not included. If $r < 0$ were also allowed, there would be a symmetric region below the axis – a shaded X instead of a shaded V.







5. Write the polar equation for the circle centered at $(x, y) = (1, 1)$ with radius $\sqrt{2}$.

   - The rectangular equation is $(x-1)^2 + (y-1)^2 = 2$ or $x^2 - 2x + y^2 - 2y = 0$. Replace x with $r\cos\theta$ and y with $r\sin\theta$. Always replace $x^2 + y^2$ with r^2. The equation becomes $r^2 = 2r\cos\theta + 2r\sin\theta$. Divide by r to get $r = 2\ (\cos\theta + \sin\theta)$.

   Note that $r = 0$ when $\theta = -\frac{\pi}{4}$. The circle goes through the origin.

6. Write the polar equations for these lines: (a) $x = 3$ (b) $y = -1$ (c) $x + 2y = 5$.

   - (a) $x = 3$ becomes $r\cos\theta = 3$ or $r = 3\sec\theta$. Remember: $r = 3\cos\theta$ is a circle.
   - (b) $y = -1$ becomes $r\sin\theta = -1$ or $r = -\csc\theta$. But $r = -\sin\theta$ is a circle.
   - (c) $x + 2y = 5$ becomes $r\cos\theta + 2r\sin\theta = 5$. Again $r = \cos\theta + 2\sin\theta$ is a circle.

Read-throughs and selected even-numbered solutions :

Polar coordinates r and θ correspond to $x = \mathbf{r\cos\theta}$ and $y = \mathbf{r\sin\theta}$. The points with $r > 0$ and $\theta = \pi$ are located **on the negative x axis**. The points with $r = 1$ and $0 \leq \theta \leq \pi$ are located on **a semicircle**. Reversing the sign of θ moves the point (x, y) to $(\mathbf{x}, -\mathbf{y})$.

Given x and y, the polar distance is $r = \sqrt{\mathbf{x^2 + y^2}}$. The tangent of θ is $\mathbf{y/x}$. The point (6,8) has $r = \mathbf{10}$ and $\theta = \mathbf{\tan^{-1}\frac{8}{6}}$. Another point with the same θ is $(\mathbf{3, 4})$. Another point with the same r is $(\mathbf{10,0})$. Another point with the same r and $\tan\theta$ is $(\mathbf{-6, -8})$.

The polar equation $r = \cos\theta$ produces a shifted **circle**. The top point is at $\theta = \pi/\mathbf{4}$, which gives $r = \mathbf{\sqrt{2}/2}$. When θ goes from 0 to 2π, we go **two** times around the graph. Rewriting as $r^2 = r\cos\theta$ leads to the xy equation $\mathbf{x^2 + y^2 = x}$. Substituting $r = \cos\theta$ into $x = r\cos\theta$ yields $x = \mathbf{\cos^2\theta}$ and similarly $y = \mathbf{\cos\theta\sin\theta}$. In this form x and y are functions of the **parameter** θ.

10 $r = 3\pi, \theta = 3\pi$ has rectangular coordinates $\mathbf{x = -3\pi, y = 0}$

16 (a) $(-1, \frac{\pi}{2})$ is the same point as $(\mathbf{1}, \mathbf{\frac{3\pi}{2}})$ or $(-1, \frac{5\pi}{2})$ or $\cdots$ (b) $(-1, \frac{3\pi}{4})$ is the same point as $(\mathbf{1}, \mathbf{\frac{7\pi}{4}})$ or $(-1, -\frac{\pi}{4})$ or $\cdots$ (c) $(1, -\frac{\pi}{2})$ is the same point as $(-1, \frac{\pi}{2})$ or $(\mathbf{1}, \mathbf{\frac{3\pi}{2}})$ or $\cdots$ (d) $r = 0, \theta = 0$ is the same point as $\mathbf{r} = \mathbf{0}$, $\theta =$ **any angle.**

18 (a) **False** ($r = 1, \theta = \frac{\pi}{4}$ is a different point from $r = -1, \theta = -\frac{\pi}{4}$) (b) **False** (for fixed r we can add any multiple of 2π to θ) (c) **True** ($r \sin\theta = 1$ is the horizontal line $y = 1$).

22 Take the line from (0,0) to (r_1, θ_1) as the base (its length is r_1). The height of the third point (r_2, θ_2), measured perpendicular to this base, is r_2 times $\sin(\theta_2 - \theta_1)$.

26 From $x = \cos^2\theta$ and $y = \sin\theta\cos\theta$, square and add to find $\mathbf{x^2 + y^2} = \cos^2\theta(\cos^2\theta + \sin^2\theta) = \cos^2\theta = \mathbf{x}$.

28 Multiply $r = a\cos\theta + b\sin\theta$ by r to find $x^2 + y^2 = ax + by$. Complete squares in $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$ and similarly in $y^2 - by$ to find $(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = (\frac{a}{2})^2 + (\frac{b}{2})^2$. This is a circle centered at $(\frac{a}{2}, \frac{b}{2})$ with radius $r = \sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2} = \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{a^2 + b^2}}$.

9.2 Polar Equations and Graphs (page 355)

The polar equation $r = F(\theta)$ is like $y = f(x)$. For each angle θ the equation tells us the distance r (which is now allowed to be negative). By connecting those points we get a polar curve. Examples are $r = 1$ and $r\cos = \theta$ (circles) and $r = 1 + \cos\theta$ (cardioid) and $r = 1/(1 + e\cos\theta)$ (parabola, hyperbola, or ellipse, depending on e). These have nice-looking polar equations – because the origin is a special point for those curves.

Note $y = \sin x$ would be a disaster in polar coordinates. Literally it becomes $r\sin\theta = \sin(r\cos\theta)$. This mixes r and θ together. It is comparable to $x^3 + xy^2 = 1$, which mixes x and y. (For mixed equations we need implicit differentiation.) Equations in this section are not mixed, they are $r = F(\theta)$ and sometimes $r^2 = F(\theta)$.

Part of drawing the picture is recognizing the symmetry. One symmetry is "through the pole." If r changes to $-r$, the equation $r^2 = F(\theta)$ stays the same – this curve has *polar symmetry.* But $r = \tan\theta$ also has polar symmetry, because $\tan\theta = \tan(\theta + \pi)$. If we go around by 180°, or π radians, we get the same result as changing r to $-r$.

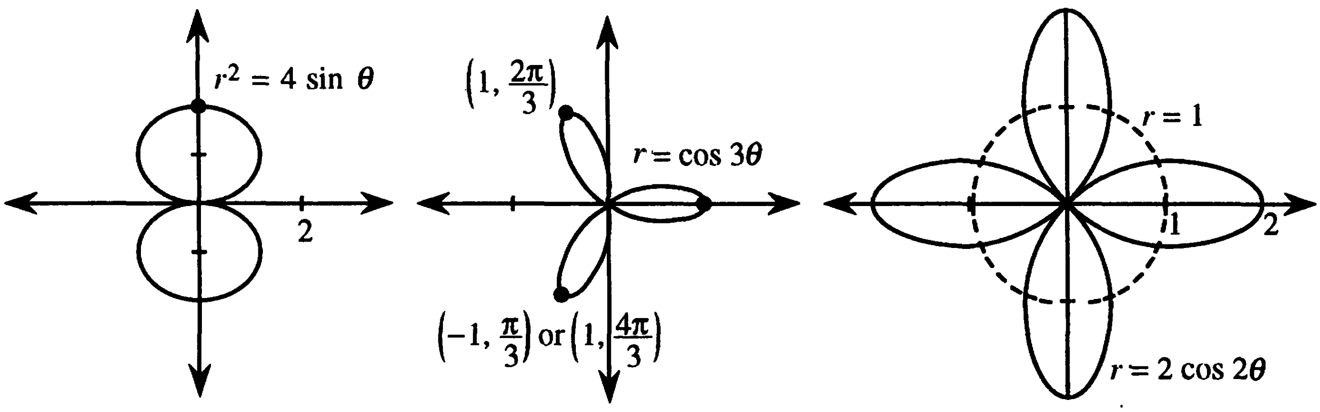
The three basic symmetries are across the x axis, across the y axis, and through the pole. Each symmetry has two main tests. (This is not clear in some texts I consulted.) Since one test could be passed without the other, I think you need to try both tests:

- x axis symmetry: θ to $-\theta$ (test 1) or θ to $\pi - \theta$ and r to $-r$ (test 2)
- y axis symmetry: θ to $\pi - \theta$ (test 1) or θ to $-\theta$ and r to $-r$ (test 2)
- polar symmetry: θ to $\pi + \theta$ (test 1) or r to $-r$ (test 2).

1. Sketch the polar curve $r^2 = 4\sin\theta$ after a check for symmetry.

   - When r is replaced by $-r$, the equation $(-r)^2 = 4\sin\theta$ is the same. This means polar symmetry (through the origin). If θ is replaced by $(\pi - \theta)$, the equation $r^2 = 4\sin(\pi - \theta) = 4\sin\theta$ is still the same. There is symmetry about the y axis. *Any two symmetries* (out of three) *imply the third.* This graph must be symmetric across the x axis. (θ to $-\theta$ doesn't show it, because $\sin\theta$ changes. But r to $-r$ and θ to $\pi - \theta$ leaves $r^2 = 4\sin\theta$ the same.) We can plot the curve in the first quadrant and reflect it to get the complete graph. Here is a table of values for the first quadrant and a sketch of the curve. The two closed parts (not circles) meet at $r = 0$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$4\sin\theta$	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4
$r = \sqrt{4\sin\theta}$	0	$\sqrt{2} \approx 1.4$	$\sqrt{2\sqrt{2}} \approx 1.7$	$\sqrt{2\sqrt{3}} \approx 1.9$	2



2. (This is Problem 9.2.9) Check $r = \cos 3\theta$ for symmetry and sketch its graph.

   - The cosine is even, $\cos(-3\theta) = \cos 3\theta$, so this curve is symmetric across the x axis (where θ goes to $-\theta$). The other symmetry tests fail. For θ up to $\frac{\pi}{2}$ we get a loop and a half in the figure. Reflection across the x axis yields the rest. The curve has three petals.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r \cos 3\theta$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0

3. Find the eight points where the four petals of $r = 2\cos 2\theta$ cross the circle $r = 1$.

   - Setting $2\cos 2\theta = 1$ leads to four crossing points $(1, \frac{\pi}{6})$, $(1, \frac{7\pi}{6})$, $(1, -\frac{\pi}{6})$, and $(1, -\frac{7\pi}{6})$. The sketch shows *four other crossing points*: $(1, \frac{\pi}{3})$, $(1, \frac{2\pi}{3})$, $(1, \frac{4\pi}{3})$ and $(1, \frac{5\pi}{3})$. These coordinates *do not* satisfy $r = 2\cos 2\theta$. But $r < 0$ yields other names $(-1, \frac{4\pi}{3})$, $(-1, \frac{5\pi}{3})$, $(-1, \frac{\pi}{3})$ and $(-1, \frac{2\pi}{3})$ for these points, that do satisfy the equation.

In general, you need a sketch to find all intersections.

4. Identify these five curves:

   (a) $r = 5\csc\theta$ (b) $r = 6\sin\theta + 4\cos\theta$ (c) $r = \frac{9}{1+6\cos\theta}$ (d) $r = \frac{4}{2+\cos\theta}$ (e) $r = \frac{1}{3-3\sin\theta}$.

   - (a) $r = \frac{5}{\sin\theta}$ is $r\sin\theta = 5$. This is the *horizontal line* $y = 5$.
   - Multiply equation (b) by r to get $r^2 = 6r\sin\theta + 4r\cos\theta$, or $x^2 + y^2 = 6y + 4x$. Complete squares to $(x-2)^2 + (y-3)^2 = 2^2 + 3^2 = 13$. This is a *circle* centered at (2,3) with radius $\sqrt{13}$.
   - (c) The pattern for conic sections (ellipse, parabola, and hyperbola) is $r = \frac{A}{1+e\cos\theta}$. Our equation has $A = 9$ and $e = 6$. The graph is a *hyperbola* with one focus at (0,0). The directrix is the line $x = \frac{9}{6} = \frac{3}{2}$.
   - (d) $r = \frac{4}{2+\cos\theta}$ doesn't exactly fit $\frac{A}{1+e\cos\theta}$ because of the 2 in the denominator. Factor it out: $\frac{2}{1+\frac{1}{2}\cos\theta}$ is an *ellipse* with $e = \frac{1}{2}$.
   - (e) $r = \frac{1}{3-3\sin\theta}$ is actually a *parabola*. To recognize the standard form, remember that $-\sin\theta = \cos(\frac{\pi}{2}+\theta)$. So $r = \frac{\frac{1}{3}}{1+\cos(\frac{\pi}{2}+\theta)}$. Since θ is replaced by $(\frac{\pi}{2}+\theta)$, the standard parabola has been rotated.

5. Find the length of the major axis (the distance between vertices) of the hyperbola $r = \frac{A}{1+e\cos\theta}$.

- Figure 9.5c in the text shows the vertices on the x axis: $\theta = 0$ gives $r = \frac{A}{1+e}$ and $\theta = \pi$ gives $r = \frac{A}{1-e}$. (The hyperbola has $A > 0$ and $e > 1$.) Notice that $(\frac{A}{1-e}, \pi)$ is on the *right* of the origin because $r = \frac{A}{1-e}$ is negative. The distance between the vertices is $\frac{A}{e-1} - \frac{A}{e+1} = \frac{2A}{e^2-1}$.

Compare with exercise 9.2.35 for the ellipse. The distance between its vertices is $2a = \frac{2A}{1-e^2}$. The distance between vertices of a parabola ($e = 1$) is $\frac{2A}{0}$ = infnty! One vertex of the parabla is out at infinity.

Read-throughs and selected even-numbered solutions :

The circle of radius 3 around the origin has polar equation $\mathbf{r = 3}$. The 45° line has polar equation $\boldsymbol{\theta = \pi/4}$. Those graphs meet at an angle of **90°**. Multiplying $r = 4\cos\theta$ by r yields the xy equation $\mathbf{x^2 + y^2 = 4x}$. Its graph is a **circle** with center at **(2,0)**. The graph of $r = 4/\cos\theta$ is the line $x = \mathbf{4}$. The equation $r^2 = \cos 2\theta$ is not changed when $\theta \to -\theta$ (symmetric across **the x axis**) and when $\theta \to \pi + \theta$ (or $r \to \mathbf{-r}$). The graph of $r = 1 + \cos\theta$ is a **cardioid**.

The graph of $r = A/(\mathbf{1} + \mathbf{e}\cos\theta)$ is a conic section with one focus at $\mathbf{(0,0)}$. It is an ellipse if $\mathbf{e < 1}$ and a hyperbola if $\mathbf{e > 1}$. The equation $r = 1/(1+\cos\theta)$ leads to $r + x = 1$ which gives a **parabola**. Then r = distance from origin equals $1 - x$ = distance from **directrix** $\mathbf{y = 1}$. The equations $r = 3(1-x)$ and $r = \frac{1}{3}(1-x)$ represent a **hyperbola** and an **ellipse**. Including a shift and rotation, conics are determined by **five** numbers.

6 $r = \frac{1}{1+2\cos\theta}$ is the hyperbola of Example 7 and Figure 9.5c: $r + 2r\cos\theta = 1$ is $r = 1 - 2x$ or $x^2 + y^2 = 1 - 4x + 4x^2$. The figure should show $r = -1$ and $\theta = \pi$ on the right branch.

14 $r = 1 - 2\sin 3\theta$ has **y axis symmetry**: change θ to $\pi - \theta$, then $\sin 3(\pi - \theta) = \sin(\pi - 3\theta) = \sin 3\theta$.

22 If $\cos\theta = \frac{r^2}{4}$ and $\cos\theta = 1 - r$ then $\frac{r^2}{4} = 1 - r$ and $r^2 + 4r - 4 = 0$. This gives $r = -2 - \sqrt{8}$ and $\mathbf{r = -2 + \sqrt{8}}$. The first r is negative and cannot equal $1 - \cos\theta$. The second gives $\cos\theta = 1 - r = 3 - \sqrt{8}$ and $\theta \approx \mathbf{80°}$ **or** $\theta \approx \mathbf{-80°}$. The curves also meet at **the origin** $\mathbf{r = 0}$ and at the point $\mathbf{r = -2}, \theta = \mathbf{0}$ which is also $\mathbf{r = +2}, \theta = \pi$.

26 The other 101 petals in $r = \cos 101\theta$ are **duplicates of the first 101**. For example $\theta = \pi$ gives $r = \cos 101\pi = -1$ which is also $\theta = 0, r = +1$. (Note that $\cos 100\pi = +1$ gives a new point.)

28 (a) Yes, x and y symmetry imply r symmetry. Reflections across the x axis and then the y axis take (x, y) to $(x, -y)$ to $(-x, -y)$ which is reflection through the origin. (b) The point $r = -1, \theta = \frac{3\pi}{2}$ satisfies the equation $r = \cos 2\theta$ and it is the same point as $r = 1, \theta = \frac{\pi}{2}$.

32 (a) $\theta = \frac{\pi}{2}$ gives $r = 1$; this is $x = 0, y = 1$ (b) The graph crosses the x axis at $\theta = 0$ and π where $x = \frac{1}{1+e}$ and $x = \frac{-1}{1-e}$. The center of the graph is halfway between at $x = \frac{1}{2}(\frac{1}{1+e} - \frac{1}{1-e}) = \frac{-e}{1-e^2}$. The second focus is twice as far from the origin at $\mathbf{\frac{-2e}{1-e^2}}$. (Check: $e = 0$ gives center of circle, $e = 1$ gives second focus of parabola at infinity.)

9.3 Slope, Length, and Area for Polar Curves (page 359)

This section does *calculus in polar coordinates*. All the calculations for $y = f(x)$ – its slope $\frac{dy}{dx}$ and area

$\int y\, dx$ and arc length $\int \sqrt{1+(\frac{dy}{dx})^2}\, dx$ – can also be done for polar curves $r = F(\theta)$. But the formulas are a little more complicated! The slope is not $\frac{dF}{d\theta}$ and the area is not $\int F(\theta)d\theta$. These problems give practice with the polar formulas for slope, area, arc length, and surface area of revolution.

1. (This is 9.3.5) Draw the 4-petaled flower $r = \cos 2\theta$ and find the area inside. The petals are along the axes.

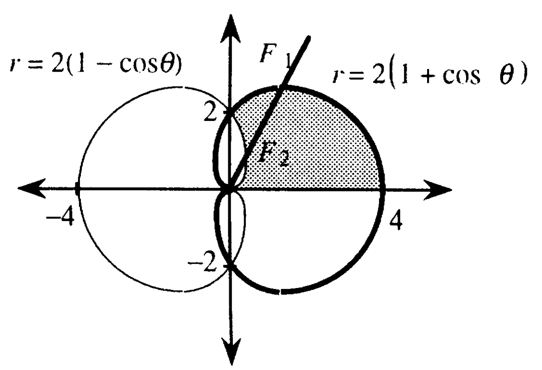
   - We compute the area of one petal and multiply by 4. The right-hand petal lies between the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$. Those are the limits of integration:

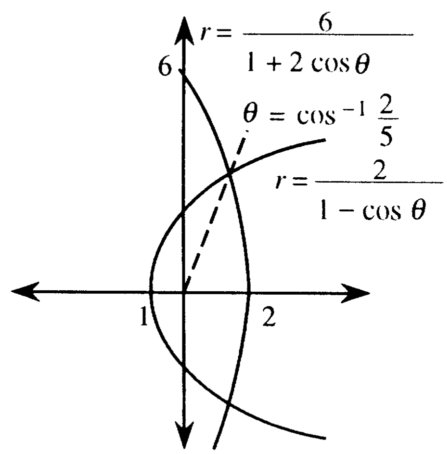
$$\text{Area} = 4\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2 d\theta = \int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)d\theta = \frac{\pi}{2}.$$

2. Find the area inside $r = 2(1+\cos\theta)$ and outside $r = 2(1-\cos\theta)$. Sketch those cardioids.

   - In the figure, half the required area is shaded. *Take advantage of symmetries!* A typical line through the origin is also sketched. Imagine this line sweeping from $\theta = 0$ to $\theta = \frac{\pi}{2}$ – the whole shaded area is covered. The outer radius is $2(1+\cos\theta)$, the inner radius is $2(1-\cos\theta)$. The shaded area is

$$\int_0^{\pi/2} \frac{1}{2}[4(1+\cos\theta)^2 - 4(1-\cos\theta)^2]d\theta = 8\int_0^{\pi/2} \cos\theta\, d\theta = 8. \qquad \text{Total area 16.}$$





3. Set up the area integral(s) between the parabola $r = \frac{2}{1-\cos\theta}$ and the hyperbola $r = \frac{6}{1+2\cos\theta}$.

   - The curves are shown in the sketch. We need to find where they cross. Solving $\frac{6}{1+2\cos\theta} = \frac{2}{1-\cos\theta}$ yields $6(1-\cos\theta) = 2(1+2\cos\theta)$ or $\cos\theta = \frac{2}{5} = .4$. At that angle $r = \frac{6}{1+2(\frac{2}{5})} = \frac{6}{1.8}$.

Imagine a ray sweeping around the origin from $\theta = 0$ to $\theta = \pi$. From $\theta = 0$ to $\theta = \cos^{-1} .4$, the ray crosses the *hyperbola*. Then it crosses the *parabola*. That is why the area must be computed in two parts. Using symmetry we find only the top half:

$$\text{Half-area} = \int_0^{\cos^{-1}.4} \frac{1}{2}(\frac{6}{1+2\cos\theta})^2 d\theta + \int_{\cos^{-1}.4}^{\pi} \frac{1}{2}(\frac{2}{1-\cos\theta})^2 d\theta.$$

Simpson's rule gives the total area (top half doubled) as approximately 12.1.

Problems 4 and 5 are about lengths of curves.

4. Find the distance around the cardioid $r = 1 + \cos\theta$.

   - Length in polar coordinates is $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$. For the cardioid this square root is

$$\sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta} = \sqrt{2 + 2\cos\theta}.$$

Half the curve is traced as θ goes from 0 to π. The total length is $\int ds = 2\int_0^\pi \sqrt{2+2\cos\theta}\, d\theta$. Evaluating this integral uses the trick $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$. Thus the cardioid length is

$$2\int_0^\pi \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 4\int_0^\pi \cos\frac{\theta}{2}\, d\theta = 8\sin\frac{\theta}{2}\Big]_0^\pi = 8.$$

5. Find the length of the spiral $r = e^{\theta/2}$ as θ goes from 0 to 2π.

   - For this curve $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$ is equal to $\sqrt{\frac{1}{4}e^\theta + e^\theta}\, d\theta = \sqrt{\frac{5}{4}e^\theta}\, d\theta = \frac{\sqrt{5}}{2}e^{\theta/2}d\theta$:

$$\text{Length } = \int_0^{2\pi} \frac{\sqrt{5}}{2}e^{\theta/2}d\theta = \sqrt{5}\, e^{\theta/2}\Big]_0^{2\pi} = \sqrt{5}(e^\pi - 1) \approx 49.5.$$

Problems 6 and 7 ask for the areas of surfaces of revolution.

6. Find the surface area when the spiral $r = e^{\theta/2}$ between $\theta = 0$ and $\theta = \pi$ is revolved about the horizontal axis.

   - From Section 8.3 we know that the area is $\int 2\pi y\, ds$. For this curve the previous problem found $ds = \frac{\sqrt{5}}{2}e^{\theta/2}d\theta$. The factor y in the area integral is $r\sin\theta = e^{\theta/2}\sin\theta$. The area is

$$\begin{aligned}\int_0^\pi 2\pi(e^{\theta/2}\sin\theta)\tfrac{\sqrt{5}}{2}e^{\theta/2}d\theta &= \sqrt{5}\pi\int_0^\pi e^\theta \sin\theta d\theta \\ &= \tfrac{\sqrt{5}\pi}{2}e^\theta(\sin\theta - \cos\theta)\Big]_0^\pi = \tfrac{\sqrt{5}\pi}{2}(e^\pi + 1) \approx 84.8.\end{aligned}$$

7. Find the surface area when the curve $r^2 = 4\sin\theta$ is revolved around the y axis.

   - The curve is drawn in Section 9.2 of this guide (Problem 1).
   - If we revolve the piece from $\theta = 0$ to $\theta = \pi/2$, and double that area, we get the total surface area. In the integral $\int_{\theta=0}^{\pi/2} 2\pi x\, ds$ we replace x by $r\cos\theta = 2\sqrt{\sin\theta}\cos\theta$. Also $ds = \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta = \sqrt{\frac{\cos^2\theta}{\sin\theta} + 4\sin\theta}\, d\theta$. The integral for surface area is not too easy:

$$4\pi\int_0^{\pi/2} 2\sqrt{\sin\theta}\cos\theta\sqrt{\frac{\cos^2\theta}{\sin\theta} + 4\sin\theta}\, d\theta = 8\pi\int_0^{\pi/2}\cos\theta\sqrt{\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= 8\pi\int_0^{\pi/2}\cos\theta\sqrt{1 + 3\sin^2\theta}\, d\theta = 8\pi\int_0^1 \sqrt{1+3u^2}\, du \text{ (where } u = \sin\theta).$$

A table of integrals gives $8\pi\sqrt{3}(\frac{u}{2}\sqrt{\frac{1}{3} + u^2} + \frac{1}{6}\ln(u + \sqrt{\frac{1}{3} + u^2})]_0^1 = 8\pi\sqrt{3}(\frac{1}{\sqrt{3}} + \frac{1}{6}\ln(2 + \sqrt{3})) \approx 34.1.$

8. Find the slope of the three-petal flower $r = \cos 3\theta$ at the tips of the petals.

   - The flower is drawn in Section 9.2. The tips are at (1,0), $(-1, \frac{\pi}{3})$, and $(-1, -\frac{\pi}{3})$. Clearly the tangent line at (1,0) is vertical (infinite slope). For the other two slopes, find $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$. From $y = r\sin\theta$ we get $\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$. Similarly $x = r\cos\theta$ gives $\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}$. Substitute $\frac{dr}{d\theta} = -3\sin 3\theta$ for this flower, and set $r = -1$, $\theta = \frac{\pi}{3}$:

$$\frac{dy}{dx} = \frac{r\cos\theta - 3\sin 3\theta\sin\theta}{-r\sin\theta - 3\sin 3\theta\cos\theta} = \frac{(-1)\cos\pi/3 - 3\sin\pi\sin\pi/3}{\sin\pi/3 - 3\sin\pi\cos\pi/3} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$

9. If $F(3) = 0$, show that the graph of $r = F(\theta)$ at $r = 0$, $\theta = 3$ has slope $\tan 3$.

   - As an example of this idea, look at the graph of $r = \cos 3\theta$ (Section 9.1 of this guide). At $\theta = \pi/6$, $\theta = \pi/2$, and $\theta = -\pi/6$ we find $r = 0$. The rays out from the origin at those three angles are tangent to the graph. In other words the slope of $r = \cos 3\theta$ at $(0, \pi/6)$ is $\tan(\pi/6)$, the slope at $(0, \pi/2)$ is $\tan(\pi/2)$ and the slope at $(0, -\pi/6)$ is $\tan(-\pi/6)$.
   - To prove the general statement, write $\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta dr/d\theta}{-r\sin\theta + \cos\theta dr/d\theta}$ as in Problem 8. With $r = F(\theta)$ and $F(3) = 0$, substitute $\theta = 3$, $r = 0$, and $dr/d\theta = F'(3)$. The slope at $\theta = 3$ is $\frac{dy}{dx} = \frac{\sin(3)F'(3)}{\cos(3)F'(3)} = \tan(3)$.

Read-throughs and selected even-numbered solutions :

A circular wedge with angle $\Delta\theta$ is a fraction $\mathbf{\Delta\theta/2\pi}$ of a whole circle. If the radius is r, the wedge area is $\mathbf{\frac{1}{2}r^2\Delta\theta}$. Then the area inside $r = F(\theta)$ is $\int \mathbf{\frac{1}{2}r^2 d\theta} = \int \mathbf{\frac{1}{2}(F(\theta))^2 d\theta}$. The area inside $r = \theta^2$ from 0 to π is $\mathbf{\pi^5/10}$. That spiral meets the circle $r = 1$ at $\theta = \mathbf{1}$. The area inside the circle and outside the spiral is $\mathbf{\frac{1}{2} - \frac{1}{10}}$. A chopped wedge of angle $\Delta\theta$ between r_1 and r_2 has area $\mathbf{\frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta}$.

The curve $r = F(\theta)$ has $x = r\cos\theta = \mathbf{F(\theta)\cos\theta}$ and $y = \mathbf{F(\theta)\sin\theta}$. The slope dy/dx is $dy/d\theta$ divided by $\mathbf{dx/d\theta}$. For length $(ds)^2 = (dx)^2 + (dy)^2 = \mathbf{(dr)^2 + (r d\theta)^2}$. The length of the spiral $r = \theta$ to $\theta = \pi$ is $\int \mathbf{\sqrt{1+\theta^2}d\theta}$. The surface area when $r = \theta$ is revolved around the x axis is $\int 2\pi y\, ds = \int \mathbf{2\pi\theta\sin\theta\sqrt{1+\theta^2}d\theta}$. The volume of that solid is $\int \pi y^2 dx = \int \mathbf{\pi\theta^2\sin^2\theta(\cos\theta - \theta\sin\theta)d\theta}$.

4 The inner loop is where $r < 0$ or $\cos\theta < -\frac{1}{2}$ or $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$. Its area is $\int \frac{r^2}{2}d\theta = \int \frac{1}{2}(1 + 4\cos\theta + 4\cos^2\theta)d\theta = [\frac{\theta}{2} + 2\sin\theta + \theta + \cos\theta\,\sin\theta]_{2\pi/3}^{4\pi/3} = \frac{\pi}{3} - 2(\sqrt{3}) + \frac{2\pi}{3} + \frac{1}{2}\sqrt{3} = \pi - \mathbf{\frac{3}{2}\sqrt{3}}$.

16 The spiral $r = e^{-\theta}$ starts at $r = 1$ and returns to the x axis at $r = e^{-2\pi}$. Then it goes inside itself (no new area). So area $= \int_0^{2\pi} \frac{1}{2}e^{-2\theta}d\theta = [-\frac{1}{4}e^{-2\theta}]_0^{2\pi} = \mathbf{\frac{1}{4}(1 - e^{-4\pi})}$.

20 Simplify $\frac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \frac{\frac{F + \tan\theta F'}{-\tan\theta F + F'} - \tan\theta}{1 + \frac{F + \tan\theta F'}{-\tan\theta F + F'}\tan\theta} = \frac{F + \tan\theta F' - \tan\theta(-\tan\theta F + F')}{-\tan\theta F + F' + \tan\theta(F + \tan\theta F')} = \frac{(1+\tan^2\theta)F}{(1+\tan^2\theta)F'} = \mathbf{\frac{F}{F'}}$.

22 $r = 1 - \cos\theta$ is the mirror image of Figure 9.4c across the y axis. By Problem 20, $\tan\psi = \frac{F}{F'} = \frac{1-\cos\theta}{\sin\theta}$. This is $\frac{\frac{1}{2}\sin^2\frac{\theta}{2}}{\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$. So $\psi = \frac{\theta}{2}$ (check at $\theta = \pi$ where $\psi = \frac{\pi}{2}$).

24 By Problem 18 $\frac{dy}{dx} = \frac{\cos\theta + \tan\theta(-\sin\theta)}{-\cos\theta\,\tan\theta - \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta(-2\sin\theta)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{6}$. At that point $x = r\,\cos\,\theta = \cos^2\frac{\pi}{6} = (\frac{\sqrt{3}}{2})^2$ and $y = r\,\sin\theta = \cos\frac{\pi}{6}\,\sin\frac{\pi}{6} = \frac{1}{2}(\frac{\sqrt{3}}{2})$. The tangent line is $\mathbf{y - \frac{\sqrt{3}}{4} = -\frac{1}{\sqrt{3}}(x - \frac{3}{4})}$.

26 $r = \sec\theta$ has $\frac{dr}{d\theta} = \sec\theta\tan\theta$ and $\frac{ds}{d\theta} = \sqrt{\sec^2\theta + \sec^2\theta\,\tan^2\theta} = \sqrt{\sec^4\theta} = \sec^2\theta$. Then arc length $= \int_0^{\pi/4}\sec^2\theta\,d\theta = \tan\frac{\pi}{4} = \mathbf{1}$. Note: $r = \sec\theta$ is the line $r\,\cos\theta = 1$ or $x = 1$ from $y = 0$ up to $y = 1$.

32 $r = 1 + \cos\,\theta$ has $\frac{ds}{d\theta} = \sqrt{(1 + 2\,\cos\,\theta + \cos^2\theta) + \sin^2\theta} = \sqrt{2 + 2\,\cos\theta}$. Also $y = r\,\sin\,\theta = (1 + \cos\,\theta)\sin\theta$. Surface area $\int 2\pi\,y\,ds = 2\pi\sqrt{2}\int_0^{\pi}(1 + \cos\,\theta)^{3/2}\sin\,\theta\,d\theta = [2\pi\sqrt{2}(-\frac{2}{5})(1 + \cos\,\theta)^{5/2}]_0^{\pi} = \frac{32\pi}{5}$.

40 The parameter θ along the ellipse $x = 4\,\cos\,\theta, y = 3\,\sin\,\theta$ is *not* the angle from the origin. For example at $\theta = \frac{\pi}{4}$ the point (x, y) is *not* on the 45° line. So the area formula $\int \frac{1}{2} r^2 d\theta$ does not apply. The correct area is 12π.

9.4 Complex Numbers (page 364)

There are two important forms for every complex number: the *rectangular form* $x+iy$ and the *polar form* $re^{i\theta}$. Converting from one to the other is like changing between rectangular and polar coordinates. In one direction use $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. In the other direction (definitely easier) use $x = r\cos\theta$ and $y = r\sin\theta$. Problem 1 goes to polar and Problem 2 goes to rectangular.

1. Convert these complex numbers to polar form: (a) $3+4i$ (b) $-5-12i$ (c) $i\sqrt{3}-1$.

   - (a) $r = \sqrt{3^2+4^2} = 5$ and $\theta = \tan^{-1}\frac{4}{3} \approx .93$. Therefore $3+4i \approx 5e^{.93i}$.
   - (b) $-5-12i$ lies in the third quadrant of the complex plane, so $\theta = \pi + \arctan^{-1}\frac{-12}{-5} \approx \pi + 1.17 \approx 4.3$. The distance from the origin is $r = \sqrt{(-5)^2+(-12)^2} = 13$. Thus $-5-12i \approx 13e^{4.3i}$.
   - (c) $i\sqrt{3}-1$ is not exactly in standard form: rewrite as $-1+i\sqrt{3}$. Then $x = -1$ and $y = \sqrt{3}$ and $r = \sqrt{1+3} = 2$. This complex number is in the second quadrant of the complex plane, since $x < 0$ and $y > 0$. The angle is $\theta = \frac{2\pi}{3}$. Then $-1+i\sqrt{3} = 2e^{2\pi/3}$.

   We chose the standard polar form, with $r > 0$ and $0 \le \theta < 2\pi$. Other polar forms are allowed. The answer for (c) could also be $2e^{(2\pi+2\pi/3)i}$ or $2e^{-4\pi i/3}$.

2. Convert these complex numbers to rectangular form: (a) $6e^{i\pi/4}$ (b) $e^{-7\pi/6}$ (c) $3e^{\pi/3}$

   - (a) The point $z = 6e^{i\pi/4}$ is 6 units out along the ray $\theta = \pi/4$. Since $x = 6\cos\frac{\pi}{4} = 3\sqrt{2}$ and $y = 6\sin\frac{\pi}{4} = 3\sqrt{2}$, the rectangular form is $3\sqrt{2}+3\sqrt{2}i$.
   - (b) We have $r = 1$. The number is $\cos(-\frac{7\pi}{6}) + i\sin(-\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.
   - (c) *There is no i in the exponent!* $3e^{\pi/3}$ is just a plain real number (approximately 8.5). Its rectangular form is $3e^{\pi/3} + 0i$.

3. For each pair of numbers find z_1+z_2 and z_1-z_2 and z_1z_2 and z_1/z_2:

   (a) $z_1 = 4-3i$ and $z_2 = 12+5i$ (b) $z_1 = 3e^{i\pi/6}$ and $z_2 = 2e^{i7\pi/4}$.

   - (a) Add $z_1+z_2 = 4-3i+12+5i = 16+2i$. Subtract $(4-3i)-(12+5i) = -8-8i$. Multiply:

   $$(4-3i)(12+5i) = 48-36i+20i-15i^2 = 63-16i$$

   To divide by $12+5i$, multiply top and bottom by its complex conjugate $12-5i$. *Then the bottom is real:*

   $$\frac{4-3i}{12+5i}\cdot\frac{12-5i}{12-5i} = \frac{33-56i}{12^2+5^2} = \frac{33}{169} - \frac{56}{169}i.$$

   - You could choose to multiply in polar form. First convert $4-3i$ to $re^{i\theta}$ with $r = 5$ and $\tan\theta = -\frac{3}{4}$. Also $12+5i$ has $r = 13$ and $\tan\theta = \frac{5}{12}$. Multiply the r's to get $5\cdot 13 = 65$. Add the θ's. This is hard without a calculator that knows $\tan^{-1}(-\frac{3}{4})$ and $\tan^{-1}(\frac{5}{12})$. Our answer is $\theta_1+\theta_2 \approx -.249$.

So multiplication gives $65e^{-.249i}$ which is close to the first answer $63 - 16i$. Probably a trig identity would give $\tan^{-1}(-\frac{3}{4}) + \tan^{-1}(\frac{5}{12}) = \tan^{-1}(-\frac{16}{63})$.

For division in polar form, divide r's and subtract angles: $\frac{5}{13}e^{i(\theta_1-\theta_2)} \approx \frac{5}{13}e^{-i}$. This is $\frac{z_1}{z_2} = \frac{5}{13}\cos(-1) + \frac{5}{13}i\sin(-1) \approx .2 - .3i \approx \frac{33}{169} - \frac{56}{169}i$.

- (b) Numbers in polar form are not easy to add. Convert to rectangular form:

$$3e^{i\pi/6} \text{ equals } 3\cos\frac{\pi}{6} + 3i\sin\frac{\pi}{6} = \frac{3\sqrt{3}}{2} + \frac{3i}{2}. \text{ Also } 2e^{i7\pi/4} \text{ equals } 2\cos\frac{7\pi}{4} + 2i\sin\frac{7\pi}{4} = \sqrt{2} - i\sqrt{2}.$$

The sum is $(\frac{3\sqrt{3}}{2} + \sqrt{2}) + (\frac{3}{2} - \sqrt{2})i$. The difference is $(\frac{3\sqrt{3}}{2} - \sqrt{2}) + (\frac{3}{2} + \sqrt{2})i$.

Multiply and divide in polar form whenever possible. *Multiply r's and add θ's*:

$$z_1 z_2 = (3 \cdot 2)e^{i(\frac{\pi}{6}+\frac{7\pi}{4})} = 6e^{\frac{23\pi i}{12}} \text{ and } \frac{z_1}{z_2} = \frac{3}{2}e^{i(\frac{\pi}{6}-\frac{7\pi}{4})} = \frac{3}{2}e^{-19\pi i/24}.$$

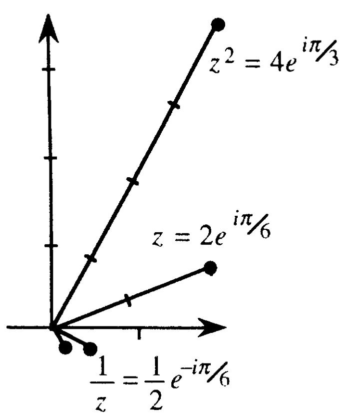
3. Find $(2 - 2\sqrt{3}i)^{10}$ in polar and rectangular form.

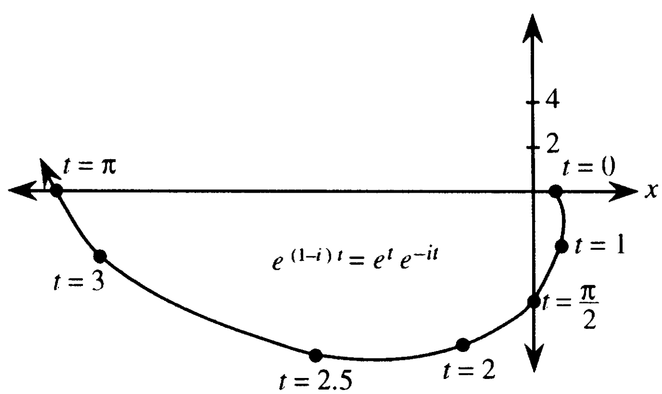
   - DeMoivre's Theorem is based on the polar form: $2 - 2\sqrt{3}i = 4e^{-i\pi/3}$. The tenth power is $(4e^{-i\pi/3})^{10} = 4^{10}e^{-10\pi/3}$. In rectangular form this is

$$4^{10}(\cos\frac{-10\pi}{3} + i\sin\frac{-10\pi}{3}) = 4^{10}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 2^{20} - (\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -2^{19} + 2^{19}i\sqrt{3}.$$

4. (This is 9.4.3) Plot $z = 2e^{i\pi/6}$ and its reciprocal $\frac{1}{z} = \frac{1}{2}e^{-i\pi/6}$ and their squares.

   - The squares are $(2e^{i\pi/6})^2 = 4e^{i\pi/3}$ and $(\frac{1}{2}e^{-i\pi/6})^2 = \frac{1}{4}e^{-i\pi/3}$. The points z, $\frac{1}{z}$, z^2, $\frac{1}{z^2}$ are plotted.





5. (This is 9.4.25) For $c = 1 - i$, sketch the path of $y = e^{ct}$ as t increases from 0.

   - The moving point e^{ct} is $e^{(1-i)t} = e^t e^{-it} = e^t(\cos(-t) + i\sin(-t))$. The table gives x and y:

t	0	.5	1.0	$\pi/2$	2.0	2.5	3.0	π
$x = e^t \cos(-t)$	1	1.4	1.5	0	−3.1	−9.8	−19.9	23.1
$y = e^t \sin(-t)$	0	−.8	−2.3	−4.8	−6.7	−7.3	−2.8	0

The sketch shows how e^{ct} spirals rapidly outwards from $e^0 = 1$.

6. For the differential equation $y'' + 4y' + 3y = 0$, find all solutions of the form $y = e^{ct}$.

   - The derivatives of $y = e^{ct}$ are $y' = ce^{ct}$ and $y'' = c^2e^{ct}$. The equation asks for $c^2e^{ct} + 4ce^{ct} + 3e^{ct} = 0$. This means that $e^{ct}(c^2 + 4c + 3) = 0$. Factor $c^2 + 4c + 3$ into $(c + 3)(c + 1)$. This is zero for $c = -3$ and $c = -1$. The pure exponential solutions are $y = e^{-3t}$ and $y = e^{-t}$. Any combination like $2e^{-3t} + 7e^{-t}$ also solves the differential equation.

7. Construct two real solutions of $y'' + 2y' + 5y = 0$. Start with solutions of the form $y = e^{ct}$.

   - Substitute $y'' = c^2e^{ct}$ and $y' = ce^{ct}$ and $y = e^{ct}$. This leads to $c^2 + 2c + 5 = 0$ or $c = -1 \pm 2i$. The pure (but complex) exponential solutions are $y = e^{(-1+2i)t}$ and $y = e^{(-1-2i)t}$. The first one is $y = e^{-t}(\cos 2t + i \sin 2t)$. The real part is $x = e^{-t} \cos 2t$; the imaginary part is $y = e^{-t} \sin 2t$. (*Note:* The imaginary part is without the i.) Each of these is a real solution, as may be checked by substitution into $y'' + 2y' + 5y = 0$.

   The other exponential is $y = e^{(-1-2i)t} = e^{-t}(\cos(-2t) + i \sin(-2t))$. Its real and imaginary parts are the *same real solutions* – except for the minus sign in $\sin(-2t) = -\sin 2t$.

Read-throughs and selected even-numbered solutions :

The complex number $3 + 4i$ has real part **3** and imaginary part **4**. Its absolute value is $r = \mathbf{5}$ and its complex conjugate is $\mathbf{3 - 4i}$. Its position in the complex plane is at $(\mathbf{3,4})$. Its polar form is $r\cos\theta + ir\sin\theta = \mathbf{re^{i\theta}}$ (**or** $\mathbf{5e^{i\theta}}$). Its square is $\mathbf{-7 - 14i}$. Its nth power is $\mathbf{r^n}e^{in\theta}$.

The sum of $1 + i$ and $1 - i$ is **2**. The product of $1 + i$ and $1 - i$ is **2**. In polar form this is $\sqrt{2}e^{i\pi/4}$ times $\mathbf{\sqrt{2}e^{-i\pi/4}}$. The quotient $(1+i)/(1-i)$ equals the imaginary number i. The number $(1+i)^8$ equals **16**. An eighth root of 1 is $w = \mathbf{(1 + i)/\sqrt{2}}$. The other eighth roots are $\mathbf{w^2, w^3, \cdots, w^7, w^8 = 1}$.

To solve $d^8y/dt^8 = y$, look for a solution of the form $y = \mathbf{e^{ct}}$. Substituting and canceling e^{ct} leads to the equation $\mathbf{c^8 = 1}$. There are **eight** choices for c, one of which is $(-1 + i)/\sqrt{2}$. With that choice $|\, e^{ct} \,| = \mathbf{e^{-t/\sqrt{2}}}$. The real solutions are Re $e^{ct} = \mathbf{e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}}$ and Im $e^{ct} = \mathbf{e^{-t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}}$.

10 $e^{ix} = i$ yields $\mathbf{x = \frac{\pi}{2}}$ (note that $\frac{i\pi}{2}$ becomes $\ln i$); $e^{ix} = e^{-1}$ yields $\mathbf{x = i}$, second solutions are $\frac{\pi}{2} + 2\pi$ and $i + 2\pi$.

14 The roots of $c^2 - 4c + 5 = 0$ must multiply to give **5**. Check: The roots are $\frac{4\pm\sqrt{16-20}}{2} = 2 \pm i$. Their product is $(2 + i)(2 - i) = 4 - i^2 = 5$.

18 The fourth roots of $re^{i\theta}$ are $r^{1/4}$ times $e^{i\theta/4}, e^{i(\theta+2\pi)/4}, e^{i(\theta+4\pi)/4}, e^{i(\theta+6\pi)/4}$. Multiply $(r^{1/4})^4$ to get r. Add angles to get $(4\theta + 12\pi)/4 = \theta + 3\pi$. The product of the 4 roots is $re^{i(\theta+3\pi)} = -re^{i\theta}$.

28 $\frac{dy}{dt} = iy$ leads to $y = e^{it} = \cos t + i \sin t$. Matching real and imaginary parts of $\frac{d}{dt}(\cos t + i \sin t) = i(\cos t + i \sin t)$ yields $\frac{d}{dt} \cos t = -\sin t$ and $\frac{d}{dt} \sin t = \cos t$.

34 Problem 30 yields $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh x$; similarly $\sin ix = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{i}{2i}(e^{-x} - e^{x}) = i \sinh x$. With $x = 1$ the cosine of i equals $\frac{1}{2}(e^{-1} + e^{1}) =$ **3.086**. The cosine of i is larger than 1!

9 Chapter Review Problems

Review Problems

R1 Express the point (r, θ) in rectangular coordinates. Express the point (a, b) in polar coordinates. Express the point (r, θ) with three other pairs of polar coordinates.

R2 As θ goes from 0 to 2π, how often do you cover the graph of $r = \cos\theta$? $r = \cos 2\theta$? $r = \cos 3\theta$?

R3 Give an example of a polar equation for each of the conic sections, including circles.

R4 How do you find the area between two polar curves $r = F(\theta)$ and $r = G(\theta)$ if $0 < F < G$?

R5 Write the polar form for ds. How is this used for surface areas of revolution?

R6 What is the polar formula for slope? Is it $dr/d\theta$ or dy/dx?

R7 Multiply $(a + ib)(c + id)$ and divide $(a + ib)/(c + id)$.

R8 Sketch the eighth roots of 1 in the complex plane. How about the roots of -1?

R9 Starting with $y = e^{ct}$, find two real solutions to $y'' + 25y = 0$.

R10 How do you test the symmetry of a polar graph? Find the symmetries of

(a) $r = 2\cos\theta + 1$ (b) $= 8\sin\theta$ (c) $r = \frac{6}{1-\cos\theta}$ (d) $r = \sin 2\theta$ (e) $r = 1 + 2\sin\theta$

Drill Problems

D1 Show that the area inside $r^2 = \sin 2\theta$ and outside $r = \frac{\sqrt{2}}{2}$ is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

D2 Find the area inside both curves $r = 2 - \cos\theta$ and $r = 3\cos\theta$.

D3 Show that the area enclosed by $r = 2\cos 3\theta$ is π.

D4 Show that the length of $r = 4\sin^3\frac{\theta}{3}$ between $\theta = 0$ and $\theta = \pi$ is $2\pi - \frac{3}{2}\sqrt{3}$.

D5 Confirm that the length of the spiral $r = 3\theta^2$ from $\theta = 0$ to $\theta = \frac{5}{3}$ is $\frac{7}{3}$.

D6 Find the slope of $r = \sin 3\theta$ at $\theta = \frac{\pi}{6}$.

D7 Find the slope of the tangent line to $r = \tan\theta$ at $(1, \frac{\pi}{2})$.

D8 Show that the slope of $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{6}$ is $\frac{2}{\sqrt{3}}$.

D9 The curve $r^2 = \cos 2\theta$ from $(1, -\frac{\pi}{4})$ to $(1, \frac{\pi}{4})$ is revolved around the y axis. Show that the surface area is $2\sqrt{2}\pi$.

D10 Sketch the parabola $r = 4/(1 + \cos\theta)$ to see its focus and vertex.

D11 Find the center of the ellipse whose polar equation is $r = \frac{6}{2-\cos\theta}$. What is the eccentricity e?

D12 The asymptotes of the hyperbola $r = \frac{6}{1+3\cos\theta}$ are the rays where $1 + 3\cos\theta = 0$. Find their slopes.

D13 Find all the sixth roots (two real, four complex) of 64.

D14 Find four roots of the equation $z^4 - 2z^2 + 4 = 0$.

D15 Add, subtract, multiply, and divide $1 + \sqrt{3}\, i$ and $1 - \sqrt{3}\, i$.

D16 Add, subtract, multiply, and divide $e^{i\pi/4}$ and $e^{-i\pi/4}$.

D17 Find all solutions of the form $y = e^{ct}$ for $y'' - y' - 2y = 0$ and $y''' - 2y' - 3y = 0$.

D18 Construct real solutions of $y'' - 4y' + 13y = 0$ from the real and imaginary parts of $y = e^{ct}$.

D19 Use a calculator or an integral to estimate the length of $r = 1 + \sin\theta$ (near 2.5?).

Graph Problems (intended to be drawn by hand)

G1 $r^2 = \sin 2\theta$

G2 $r = 6\sin\theta$

G3 $r = \sin 4\theta$

G4 $r = 5\sec\theta$

G5 $r = e^{\theta/2}$

G6 $r = 2 - 3\cos\theta$

G7 $r = \frac{6}{1+2\cos\theta}$

G8 $r = \frac{1}{1-\sin\theta}$