Unit 5: Linear Equations with Constant Coefficients

2.5.1

Letting

$$y = e^{rx} \tag{1}$$

we have

$$y' = re^{rx} \tag{2}$$

and

$$y'' = r^2e^{rx}. \tag{3}$$

a. Using (1), (2), and (3) in

$$y'' - 9y' - 36y = 0 \tag{4}$$

yields

$$r^2e^{rx} - 9re^{rx} - 36e^{rx} = 0,$$

or

$$e^{rx}(r^2 - 9r - 36) = 0,$$

or

$$e^{rx}(r - 12)(r + 3) = 0. \tag{5}$$

Since $e^{rx} \neq 0$, we see that (5) is satisfied only if $r = 12$ or $r = -3$. Hence, $y = e^{12x}$ and $y = e^{-3x}$ are solutions of (1), so that the general solution of (4) is

$$y = c_1e^{12x} + c_2e^{-3x}. \tag{6}$$

2.5.1 continued

b. Using (1), (2), and (3) in

$$y'' - 12y' + 36y = 0 \tag{7}$$

we obtain

$$e^{rx}(r^2 - 12r + 36) = 0$$

or

$$e^{rx}(r - 6)^2 = 0. \tag{8}$$

Thus, $r = 6$ is a repeated root of $(r - 6)^2 = 0$; hence, two linearly independent solutions of (7) are $y = e^{6x}$ and $y = xe^{6x}$. We, therefore, conclude that the general solution of (7) is

$$y = c_1e^{6x} + c_2xe^{6x}$$

or

$$y = (c_1 + c_2x)e^{6x}. \tag{9}$$

c. Again using (1), (2), and (3)

$$y'' - 8y' + 25y = 0 \tag{10}$$

becomes

$$e^{rx}(r^2 - 8r + 25) = 0. \tag{11}$$

Thus, (11) is satisfied if and only if

$$r^2 - 8r + 25 = 0. \tag{12}$$

By the quadratic formula, we conclude from (12) that

2.5.1 continued

$$r = \frac{8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

so that either

$$r = 4 + 3i \tag{13}$$

or

$$r = 4 - 3i. \tag{14}$$

Using (13), we have that $y = e^{(4 + 3i)x}$ satisfies equation (10). That is, letting $L(y) = y'' - 8y' + 25y$

$$L[e^{(4 + 3i)x}] = 0. \tag{15}$$

Now

$$\begin{aligned} e^{(4 + 3i)x} &= e^{4x + i3x} \\ &= e^{4x}e^{i3x} \\ &= e^{4x}(\cos 3x + i \sin 3x) \\ &= e^{4x}\cos 3x + ie^{4x}\sin 3x. \end{aligned} \tag{16}$$

Equation (16) shows that the real and imaginary parts of $e^{(4 + 3i)x}$ are $e^{4x}\cos 3x$ and $e^{4x}\sin 3x$ respectively we have from the theory of the lecture [i.e., $L(u + iv) = 0 \leftrightarrow L(u) = L(v) = 0$] that

$$y = e^{4x}\cos 3x \tag{17}$$

and

$$y = e^{4x}\sin 3x \tag{18}$$

2.5.1 continued

are both solutions of (10). Since the quotient of $e^{4x}\cos 3x$ and $e^{4x}\sin 3x$ is not constant, (17) and (18) are linearly independent, so we may conclude that

$$y = c_1e^{4x}\cos 3x + c_2e^{4x}\sin 3x,$$

or

$$y = e^{4x}(c_1\cos 3x + c_2\sin 3x) \tag{19}$$

is the general solution of (10).

[As for (14), which we have thus far neglected, it would lead to the solutions $e^{4x}\cos x$ and $-e^{4x}\sin x$ (i.e., the real and imaginary parts of $e^{(4 - 3i)x}$). These solutions are contained in (19); the first with $c_1 = 1$, $c_2 = 0$, and the second with $c_1 = 0$, $c_2 = -1$. Thus, it is sufficient to work with either (13) or (14).]

2.5.2(L)

On the surface, it would appear that the main aim of this exercise is to reinforce the computational techniques described in the previous exercise, our text, and in the lecture. While we certainly want to do this, there are two other very important concepts that we wish to introduce through this exercise.

First of all, we would like to get you used to thinking in terms of $L(e^{rx})$ rather than consciously substituting e^{rx} for y and then solving mechanically for the roots of the resulting equation. One reason for doing this is to emphasize that all solutions of $L(y) = 0$ have the <u>basic</u>* form $y = e^{rx}$, i.e., $L(e^{rx}) = 0$ may be thought of as being an equation involving <u>two</u> unknowns,** r and x, and there are always values of r that satisfy this equation <u>provided the equation has constant coefficients</u>.

*We shall explain what we mean by "basic" in part (b).

**x is still the usual independent variable, but r is thought of as being a parameter meaning that we may compute $L(e^{rx})$ for different values of r, but once r is chosen it remains fixed in the expression e^{rx}.

2.5.2(L) continued

Secondly, and we shall do this as a note at the end of this exercise, we want to emphasize why, in a manner of speaking, we have lost no generality in restricting our study of this unit to equations of order 2.

a. If we simply imitate the technique of the lecture, we let

$$y = e^{rx} \tag{1}$$

from which we obtain

$$y' = re^{rx} \tag{2}$$

and

$$y'' = r^2e^{rx}. \tag{3}$$

If we use the results of (1), (2), and (3) in

$$y'' + 2y' - 3y = 0 \tag{4}$$

we obtain

$$r^2e^{rx} + 2re^{rx} - 3e^{rx} = 0$$

or

$$e^{rx}(r^2 + 2r - 3) = 0. \tag{5}$$

Since $e^{rx} \neq 0$, we see from (5) that

$$r^2 + 2r - 3 = 0, \tag{5'}$$

from which it follows that

$$r = -3 \quad \text{or} \quad r = 1. \tag{6}$$

From (6) we see that the roots of (5') are real and distinct, so the general solution of (4) is

2.5.2(L) continued

$$y = c_1e^{-3x} + c_2e^x. \tag{7}$$

Actually, we solve problems like this in the previous unit so that you could get adjusted to the technique as soon as possible. What we want to say now is that we may view (4) as

$$L(y) = 0$$

where

$$L(y) = y'' + 2y' - 3y. \tag{8}$$

Notice that (8) makes sense even though it is <u>not</u> a <u>differential equation</u>. (It becomes an equation only when we <u>equate</u> (8) with some function of x.)

So, just as we did in Exercise 2.4.1, we may compute $L(u)$, from (8), where u is any twice-differentiable function of x.

In particular, using (1), (2), and (3), we see that with y (or u) $= e^{rx}$ where r is a fixed but arbitrary constant

$$L(e^{rx}) = e^{rx}(r^2 + 2r - 3).* \tag{9}$$

The key point of (9) is that it shows us that we may view $L(e^{rx})$ as a function of r. That is, in (9) we may hold x fixed and see how $L(e^{rx})$ varies with r. In other words, and this shall become very important, for example, in Exercise 2.5.6, it makes sense to talk about such expressions as $\frac{\partial L(e^{rx})}{\partial r}$. [In fact, with $L(e^{rx})$ as in (9), we may use the product rule to obtain:

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*Quite in general, if $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_2y'' + a_1y' + a_oy$ where $a_o, \ldots, a_n$ are <u>constants</u>,

$$L(e^{rx}) = e^{rx}(r^n + a_{n-1}r^{n-1} + \ldots + a_2r^2 + a_1r + a_o).$$

That is, to compute $L(e^{rx})$, we write e^{rx} as one factor, and obtain the other factor by replacing y by r in L(y) and the derivative by an exponent (where it is understood that y is the "0^{th}" derivative of y with respect to x).

2.5.2(L) continued

$$\frac{\partial L(e^{rx})}{\partial r} = e^{rx}(2r + 2) + xe^{rx*}(r^2 + 2r - 3)$$

$$= e^{rx}(2r + 2 + xr^2 + 2rx - 3x).]$$

Returning to our main problem, we see from (9) that $L(e^{rx}) = 0 \leftrightarrow r = -3$ or $r = 1$ so as far as solving this exercise is concerned, setting (9) equal to 0 is simply a compact form for how we solved the equation previously.

b. To emphasize the role of equation (9), we see that

$$y'' - 14y' + 49y = 0 \tag{10}$$

is

$$L(y) = 0$$

where

$$L(y) = y'' - 14y' + 49y. \tag{11}$$

From (11), we conclude that

$$L(e^{rx}) = e^{rx}(r^2 - 14r + 49)$$

$$= e^{rx}(r - 7)^2. \tag{12}$$

We conclude from (12) that

$$L(e^{rx}) = 0 \leftrightarrow r = 7$$

so that

$$y = e^{7x}$$

is the <u>only</u> solution of (10) <u>of the form $y = e^{rx}$</u>.

*Remember that we are differentiating with respect to r so x is the "constant" in e^{rx}.

2.5.2(L) continued

Thus, the 1-parameter family $y = ce^{7x}$ is a solution of (10), but if we want the general solution of (10), we need to find another solution which is not linearly dependent with e^{7x}. As shown in the lecture, one such other solution is xe^{7x}.

What we would like to do here is derive the fact that $y = xe^{7x}$ is a solution of (10) in a way which is much different from our lecture procedure but which works more efficiently than our classroom procedure for higher order equations.

The key to the new technique is that if

$$L(y) = 0$$

has constant coefficients, then

$$\frac{\partial L(e^{rx})}{\partial r} = L\left[\frac{\partial}{\partial r}(e^{rx})\right]. \tag{13}$$

(Again, the proof is saved for an optional exercise.)

What (13) says is that if $L(y)$ has <u>constant coefficients</u>, we may compute $\frac{\partial L(e^{rx})}{\partial r}$ by moving inside the parentheses to differentiate. The significance of this is that since

$$\frac{\partial}{\partial r}(e^{rx}) = xe^{rx}$$

equation (13) tells us that

$$\frac{\partial L(e^{rx})}{\partial r} = L(xe^{rx}). \tag{14}$$

Equation (14) is very powerful. In fact, if we now return to (12) we have from (12) that

$$\begin{aligned}\frac{\partial L(e^{rx})}{\partial r} &= e^{rx}2(r - 7) + xe^{rx}(r - 7)^2 \\ &= (r - 7)e^{rx}[2 + x(r - 7)]. \end{aligned} \tag{15}$$

2.5.2(L) continued

From (14), we may replace $\frac{\partial L(e^{rx})}{\partial r}$ by $L(xe^{rx})$ so that (15) becomes

$$L(xe^{rx}) = (r - 7)e^{rx}(2 + xr - 7x). \tag{16}$$

Since $r - 7 = 0$ when $r = 7$, we may let $r = 7$ in (16) to obtain

$$L(xe^{rx}) = 0,$$

and, thus $y = xe^{7x}$ satisfies $L(y) = 0$.

The fact that e^{7x} and xe^{7x} are linearly independent follows from the fact that their ratio is non-constant.

Hence, we have shown that the general solution of (10) is

$$y = c_1e^{7x} + c_2xe^{7x}$$

or

$$y = (c_1 + c_2x)e^{7x}.$$

A NOTE ON SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Up to now, we have been concentrating on the second order linear equation. Suppose we now look at

$$L(y) = 0 \tag{1}$$

where

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_oy.$$

Letting $y = e^{rx}$, it is not difficult to see that equation (1) becomes

$$e^{rx}(r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_o) = 0,$$

and since e^{rx} is never zero, we see that $L(e^{rx}) = 0$ if and only if r satisfies the equation:

$$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_o = 0. \tag{2}$$

Now (2) is a polynomial equation with real coefficients.

In the previous block we saw that every polynomial with complex coefficients factored linearly. Unfortunately, the same result does not hold for real coefficients. For example,

$$x^2 + 1$$

has no linear factor as we restrict our coefficients to the real numbers, but with respect to the complex numbers, it factors linearly into $(x + i)(x - i)$.

What is nice, however, is that even with real coefficients, the only irreducible (unfactorable) polynomials, other than linear ones, are quadratics. This follows from the fact that since (2) has real coefficients, the complex conjugate of any root of (2) is also a root of (2). What this means is that if $r = r_1$ is a root of (2) [so that $(r - r_1)$ is a factor of the left side of (2)] and r_1 is not real, then the complex conjugate, $\overline{r}_1$, of r_1 is unequal to r_1* and is also a root of (2).

In other words, if $(r - r_1)$ is a factor of

$$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_o \tag{3}$$

and r_1 is non-real, then $\overline{r}_1 \neq r_1$, and $(r - \overline{r}_1)$ is also a factor of (3). Since $r - r_1$ and $r - \overline{r}_1$ are different factors of (3), their product is also a factor of (3).

Hence, (3) is divisible by

$$\begin{aligned}(r - r_1)(r - \overline{r}_1) &= r^2 - (r_1 + \overline{r}_1)r + r_1\overline{r}_1 \\ &= r^2 - 2\mathrm{Re}(r_1)r + |r_1|^2 \end{aligned} \tag{4}$$

*Recall from Block 1 that $z = \overline{z} \leftrightarrow z$ is real (i.e., $a + bi = a - bi \leftrightarrow 2bi = 0 \leftrightarrow b = 0$).

but since $\mathrm{Re}(r_1)$ and $|r_1|^2$ are real even when r_1 is non-real, we see from (4) that r_1 and $\bar{r}_1$ are roots of the real quadratic equation

$$r^2 - 2\mathrm{Re}(r_1) + |r_1|^2 = 0. \tag{5}$$

In summary, then, if $r = r_1$ is any (complex) root of (2), then either $(r - r_1)$ is a real factor of (3) if r_1 is real, or $r^2 - 2\mathrm{Re}(r_1) + |r_1|^2$ is a real factor of (3) if r_1 is non-real.

[Notice that for higher degree equations, we may not know how to solve for the roots of (2), or, equivalently, the factors of (3), but whatever the factors are, the only irreducible ones are those of first or second degree (linear or quadratic).]

Hence, in theory, at least given that the equation $L(y) = 0$ is an nth order linear differential equation with constant real coefficients, the problem of finding solutions is essentially no worse than solving nth order equations.

The following optional exercise is designed to make the remarks in this note more concrete, but the student who wishes to ignore this note at this point is free to do so.

2.5.3 (Optional)

We know that $r_1 = 3$, $r_2 = 4 + 3i$, and $r_3 = 5 - 2i$ are roots of a 5th degree polynomial equation with real coefficients. Since non-real roots of such polynomial equations occur in pairs of complex conjugates we see that $r = 4 - 3i$ (since $4 - 3i$ is the complex conjugate of the non-real root $4 + 3i$) and $r = 5 + 2i$ are also roots of our fifth-degree polynomial equation.

Since a fifth-degree polynomial equation can't have more than five roots, the five roots $r_1 = 3$, $r_2 = 4 + 3i$, $\bar{r}_2 = 4 - 3i$, $r_3 = 5 - 2i$, and $\bar{r}_3 = 5 + 2i$ are the only roots of $P_5(r) = 0$. Hence, the linear factors (including those with non-real coefficients) of $P_5(r)$ are $(r - 3)$, $(r - [4 + 3i])$, $(r - [4 - 3i])$, $(r - [5 - 2i])$, and $(r - [5 + 2i])$ [where we are using the usual result that in any polynomial equation

$$P(x) = 0 \tag{1}$$

2.5.3 continued

$x = x_o$ is a root of (1) $\leftrightarrow$ $(x - x_o)$ is a factor of $P(x)$.]

So that <u>up to a constant multiple</u> [i.e., the leading coefficient of $P_5(r)$],

$$P_5(r) = (r - 3)\{(r - [4 + 3i])(r - [4 - 3i])\}$$

$$\{(r - [5 - 2i])(r - [5 + 2i])\}. \qquad (2)$$

The key now is that

$$(r - [4 + 3i])(r - [4 - 3i]) = r^2 - [4 + 3i + 4 - 3i]r + 16 - 9i^2$$

$$= r^2 - 8r + 25.* \qquad (3)$$

Similarly,

$$(r - [5 - 2i])(r - [5 + 2i]) = r^2 - 2\mathrm{Re}(5 - 2i)r + |5 - 2i|^2$$

$$= r^2 - 10r + 29. \qquad (4)$$

If we now assume that the coefficient of r^5 is 1 in $P_5(r)$, putting (3) and (4) into (2) yields

$$P_5(r) = (r - 3)(r^2 - 8r + 25)(r^2 - 10r + 29).** \qquad (5)$$

It is needless busy work to expand the right side of (5), but it should be noticed that the two real quadratic factors of $P_5(r)$ in (5) are irreducible if we insist on real coefficients.

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*With $r_2 = 4 + 3i$ and $\overline{r}_2 = 4 - 3i$, $2\mathrm{Re}(r_2) = 8$ and $|r_2|^2 = 4^2 + 3^2 = 25$. Hence, (3) agrees with the general more abstract result of equation (4) in the note at the end of the previous exercise.

**Had the coefficient of r^5 in $P_5(r)$ been m, then (5) would be replaced by

$$P_5(r) = m(r - 3)(r^2 - 8r + 25)(r^2 - 10r + 29).$$

2.5.4

Given that

$$L(y) = \frac{d^5y}{dx^5} - 2\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0 \tag{1}$$

we have that

$$\begin{aligned} L(e^{rx}) &= e^{rx}(r^5 - 2r^3 + r) \\ &= e^{rx}r(r^4 - 2r^2 + 1) \\ &= e^{rx}r(r^2 - 1)^2 \\ &= e^{rx}r(r + 1)^2(r - 1)^2. \end{aligned} \tag{2}$$

From (2), we see that

$$L(e^{rx}) = 0 \leftrightarrow r = 0,\ r = -1,\ \text{or } r = 1,$$

so that $y = e^{ox} = 1$, $y = e^{-x}$, and $y = e^x$ are three <u>linearly independent</u> solutions of (1).

Since $r = 1$ and $r = -1$ are double roots of $r(r + 1)^2(r - 1)^2 = 0$, we have that $y = xe^x$ and xe^{-x} are two additional linear independent solutions of (1). Hence, the general solution of (1) is given by

$$y = c_1e^o + c_2e^x + c_3xe^x + c_4e^{-x} + c_5xe^{-x}$$

or

$$y = c_1 + (c_2 + c_3x)e^x + (c_4 + c_5x)e^{-x}$$

is the general solution of (1).

2.5.5

Since the curve satisfies the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0 \tag{1}$$

we see that the solutions of (1) of the form $y = e^{rx}$ are determined by the polynomial equation

$$r^4 - 2r^3 + 2r^2 = 0$$

or

$$r^2(r^2 - 2r + 2) = 0. \tag{2}$$

$r = 0$ is a double root of (2); hence,

$$y = e^{0x} = 1 \quad \text{and} \quad y = xe^{0x} = x \tag{3}$$

are solutions of (1).

The other roots of (2) are given by

$$r = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i.$$

Hence, the real and imaginary parts of $e^{(1 + i)x}$ are also solutions of (2).

Since

$$e^{(1 + i)x} = e^x e^{ix}$$

$$= e^x(\cos x + i \sin x)$$

this means that

$$y = e^x \cos x \quad \text{and} \quad y = e^x \sin x \tag{4}$$

are also solutions of (1).

2.5.5 continued

Combining the results of (3) and (4), we see that

$$y = c_1 + c_2x + c_3e^x\cos x + c_4e^x\sin x \tag{5}$$

is the general solution of (1).

From (5), we see that

$$y' = c_2 + c_3[e^x\cos x - e^x\sin x] + c_4[e^x\sin x + e^x\cos x] \tag{6}$$

$$y'' = c_3[e^x\cos x - e^x\sin x - e^x\sin x - e^x\cos x]$$

$$+ c_4[e^x\sin x + e^x \cos x + e^x\cos x - e^x\sin x]$$

or

$$y'' = -2c_3e^x\sin x + 2c_4e^x\cos x, \tag{7}$$

and

$$y''' = -2c_3[e^x\sin x + e^x\cos x] + 2c_4[e^x\cos x - e^x\sin x]. \tag{8}$$

Assuming that $y = y_0$, $y' = y_0'$, $y'' = y_0''$, and $y''' = y_0'''$ when $x = 0$, equations (5), (6), (7), and (8) yield the linear algebraic system

$$\left.\begin{array}{llllll} y_0 & = c_1 & & + c_3 & & \\ y_0' & = & c_2 & + c_3 & + c_4 & \\ y_0'' & = & & & 2c_4 & \\ y_0''' & = & & -2c_3 & + 2c_4 & \end{array}\right\} \tag{9}$$

Equation (9) should be emphasized with respect to the meaning of "general solution." In particular, at least at $x = 0$, the system (9) may be solved uniquely to determine c_1, c_2, c_3, c_4 for any given values of y_0, y_0', y_0'', y_0'''. In fact, the determinant of coefficients in (9) is

2.5.5 continued

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & -2 & 2 \end{vmatrix}^{*} = \begin{vmatrix} 0 & 2 \\ -2 & 2 \end{vmatrix} = 4 \neq 0.$$

In this exercise, we are told that

$y_0 = 2$, $y_0' = 0$, $y_0'' = 2$, and $y_0''' = 0$.

Hence, (9) becomes

$$\left.\begin{array}{ll} \text{(i)} & 2 = c_1 + c_3 \\ \text{(ii)} & 0 = c_2 + c_3 + c_4 \\ \text{(iii)} & 2 = 2c_4 \\ \text{(iv)} & 0 = -2c_3 + 2c_4 \end{array}\right\} \tag{10}$$

From (iii),

$c_4 = 1$

and with $c_4 = 1$, (iv) implies $c_3 = 1$.

With $c_3 = c_4 = 1$, we see from (ii) that $c_2 = -2$; and from (i) that $2 = c_1 + 1$, so that $c_1 = 1$. Thus, the curve in question has (5) as its equation with $c_1 = c_3 = c_4 = 1$ and $c_2 = -2$. That is,

$$y = 1 - 2x + e^x \cos x + e^x \sin x. \tag{11}$$

By the fundamental existence theorem, there can be no other solution of (1) which satisfies the given initial conditions.

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*For those who may be a bit weak on determinants, have patience until we treat this topic in Block 3.

2.5.6(L)

Here we wish to establish the following important result.

Suppose that $L(y) = 0$ is an nth order linear differential equation with constant coefficients. Suppose also that $r = r_1$ is an m-fold root of $L(e^{rx}) = 0$, meaning that $L(e^{rx})$ is divisible by $(r - r_1)^m$, but not by $(r - r_1)^{m+1}$. [For example, if $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y$, we are saying that $L(e^{rx}) = e^{rx}(r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0)$ and that $(r - r_1)^m$ is a factor of $r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0$ but $(r - r_1)^{m+1}$ isn't.]

Our technique extends the method given in Exercise 2.5.2(L). Namely, it turns out that

$$\frac{\partial L(e^{rx})}{\partial r} = L\left[\frac{\partial (e^{rx})}{\partial r}\right]$$

may be generalized to include

$$\frac{\partial^k L(e^{rx})}{\partial r^k} = L\left[\frac{\partial^k (e^{rx})}{\partial r^k}\right].$$

Thus, in the given exercise, we have

$$L(y) = y''' - 3y'' + 3y' - y = 0. \tag{1}$$

Hence

$$L(e^{rx}) = e^{rx}(r^3 - 3r^2 + 3r - 1)$$

or

$$L(e^{rx}) = e^{rx}(r - 1)^3. \tag{2}$$

From (3), we see that

$$L(e^{rx}) = 0 \leftrightarrow r = 1. \tag{4}$$

2.5.6(L) continued

Hence

$$y = e^x \tag{5}$$

is a solution of (1) and because $r = 1$ is a 3-fold root of (2), we might expect by "intuitive" induction that from (5) we may also conclude that

$$y = xe^x \quad \text{and} \quad y = x^2e^x \tag{6}$$

are also (linearly independent) solutions of (1).

We may verify that our conjecture in (6) is correct by taking the partial derivative of (2) with respect to r to obtain

$$\begin{aligned}\frac{\partial L(e^{rx})}{\partial r} &= e^{rx}[3(r-1)^2] + xe^{rx}(r-1)^3 \\ &= (r-1)^2e^{rx}[3 + x + (r-1)] \\ &= (r-1)^2e^{rx}(2 + x + r).\end{aligned} \tag{7}$$

Since

$$\frac{\partial(Le^{rx})}{\partial r} = L\left[\frac{\partial(e^{rx})}{\partial r}\right] = L(xe^{rx}),$$

we have from (7) that

$$L(xe^{rx}) = (r-1)^2e^{rx}(2 + x + r), \tag{8}$$

and letting $r = 1$ in (8) yields,

$$L(xe^x) = 0$$

so that $y = xe^x$ is also a solution of (1).

If we now return to (7) and again take the partial derivative with respect to r, we obtain

2.5.6(L) continued

$$\frac{\partial^2 L(e^{rx})}{\partial r^2} = 2(r - 1)e^{rx}(2 + x + r) + (r - 1)^2 xe^{rx}(2 + x + r) + $$
$$+ (r - 1)^2 e^{rx*}$$
$$= (r - 1)e^{rx}[2(2 + x + r) + (r - 1)x(2 + x + r) +$$
$$+ (r - 1)]. \tag{9}$$

The important point is that

(i) the right side of (9) is 0 when $r = 1$

and

(ii) the left side of (9) is equal to

$$L\left[\frac{\partial(e^{rx})}{\partial r^2}\right] = L[x^2 e^{rx}].$$

Using (i) and (ii) in (9) yields

$$L(x^2 e^x) = 0$$

so that $y = x^2 e^x$ is also a solution of (1). Hence, since $\{e^x, xe^x, x^2 e^x\}$ is linearly independent and each member is a solution of the third order linear differential equation (1), the general solution of (1) is

$$y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x$$

or

$$y = (c_1 + c_2 x + c_3 x^2)e^x.$$

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*Where we have observed that the right side of (7) is a product of three factors involving r, and accordingly we used the product rule for the product of three functions of r (wherein we write the product three times, each time differentiating a different factor).

2.5.6(L) continued

Note

This exercise, together with our note at the end of Exercise 2.5.2, tells us the form of every possible solution of the general nth order linear differential equation with constant coefficients

$$L(y) = 0. \qquad (1)$$

Namely,

(1) We look at $L(e^{rx})$ to find the values of r for which $L(e^{rx}) = 0$.

(2) If $r = r_1$ yields a solution of $L(e^{rx}) = 0$ and r_1 is real then, of course, $y = e^{r_1 x}$ is a real solution of (1).

(3) If $r = r_1$ yields a solution of $L(e^{rx}) = 0$ and r_1 is non-real then $L(e^{rx})$ has the real quadratic factor $(r - r_1)(r - \bar{r}_1) = r^2 - 2\text{Re}(r_1)r + |r_1|^2$, in which case $y = e^{\alpha x}\sin \beta x$ and $y = e^{\alpha x}\sin \beta x$ (where $r_1 = \alpha + i\beta$) are both solutions of (1).

(4) If r_1 is an m-fold root of $L(e^{rx}) = 0$ then in addition to $y = e^{r_1 x}$ we also have $y = xe^{r_1 x}$, $y = x^2 e^{r_1 x}$, ..., and $y = x^{m-1}e^{r_1 x}$ as linearly independent solutions of (1).

In summary, then,

(A) If r_1 is real and an m-fold root of $L(e^{rx}) = 0$, then

$$e^{r_1 x},\ xe^{r_1 x},\ \ldots,\ x^{m-1}e^{r_1 x}$$

are all linearly independent solutions of (1). In particular, then, if $r_1 = 0$ these solutions (since $e^{0x} = 1$) take the form

$$y = 1,\ y = x,\ y = x^2,\ \ldots,\ \text{and } y = x^{m-1}.$$

Thus, if r_1 is real our solutions can only be of the form

$$y = x^k e^{r_1 x}, \qquad (10)$$

2.5.6(L) continued

[where (10) takes the form $y = x^k$ if $r_1 = 0$].

(B) If r_1 is non-real and an m-fold root of $L(e^{rx}) = 0$, then the solutions still have the form

$$y = e^{r_1 x},\ y = xe^{r_1 x},\ \ldots,\ y = x^{m-1}e^{r_1 x}$$

but are now non-real.

If we write $r_1 = \alpha_1 + i\beta_1$, these solutions become

$$\left.\begin{aligned}
&y = e^{(\alpha_1 + i\beta_1)x} = e^{\alpha_1 x}(\cos \beta_1 x + i \sin \beta_1 x)\\
&y = xe^{(\alpha_1 + i\beta_1)x} = xe^{\alpha_1 x}(\cos \beta_1 x + i \sin \beta_1 x)\\
&\vdots\\
&\text{and}\\
&y = x^{m-1}e^{(\alpha_1 + i\beta_1)x} = x^{m-1}e^{\alpha_1 x}(\cos \beta_1 x + i \sin \beta_1 x)
\end{aligned}\right\} \quad (11)$$

Remembering that $L(u + iv) = 0 \leftrightarrow L(u) = L(v) = 0$, we conclude from (11) that the only possible <u>real</u> solutions in this case have the form

$$e^{\alpha_1 x}\cos \beta_1 x,\ e^{\alpha_1 x}\sin \beta_1 x,\ xe^{\alpha_1 x}\cos \beta_1 x,\ xe^{\alpha_1 x}\sin \beta_1 x,$$

$$x^2e^{\alpha_1 x}\cos \beta_1 x,\ x^2e^{\alpha_1 x}\cos \beta_1 x,\ \text{etc.}$$

In other words, only functions of the form

$$\left.\begin{aligned}
&y = x^k e^{\alpha_1 x}\sin \beta_1 x\\
&\text{and}\\
&y = x^k e^{\alpha_1 x}\cos \beta_1 x
\end{aligned}\right\} \quad \text{where } k \text{ is a \underline{whole number} and } \alpha_1 \text{ and } \beta_1 \text{ are real}$$

2.5.6(L) continued

can be solutions. In the case that r_1 is real, $\beta_1 = 0$; hence, $y = x^k e^{\alpha_1 x}\cos \beta_1 x$ reduces to

$$y = x^k e^{\alpha_1 x}$$

which is also the solution found in (8).

Thus, the most general <u>single</u> term which can be a solution of (1) is

$$\left.\begin{array}{l} y = x^k e^{\alpha_1 x}\sin \beta_1 x \\ \text{or} \\ y = x^k e^{\alpha_1 x}\cos \beta_1 x \end{array}\right\} \begin{array}{c} k\ \underline{\text{whole}} \\ \alpha_1,\ \beta_1\ \underline{\text{real}} \end{array} \qquad (12)$$

Hence, the general solution of (1) consists of appropriate linear combinations of terms of the form (12).

As special cases, (12) contains the solutions $y = x^k$, $y = e^{\alpha_1 x}$, $y = \sin \beta_1 x$, $y = \cos \beta_1 x$, $y = e^{\alpha_1 x}\cos \beta_1 x$, etc., but nothing other than linear combinations of terms which have the specific form (12) can be solutions of $L(y) = 0$, provided the equation has constant coefficients.

For example, $y = \sqrt{x}$ cannot be a solution of any linear differential equation with constant coefficients since there is <u>no</u> way of picking a whole number k and real numbers α_1 and β_1 so that <u>any</u> linear combination of the terms having the form (12) can equal $x^{\frac{1}{2}}$.

It is in this sense that we may say that we know everything there is to know about solutions of $L(y) = 0$ when the equation has constant coefficients.

2.5.7 (Optional)

We want $y = xe^{2x}\cos 3x$ to be a solution of

$$L(y) = 0 \tag{1}$$

where (1) has constant coefficients.

We know that $y = e^{2x}\cos 3x$ can be a solution of (1) if and only if $y = e^{(2 + 3i)x}$ is a solution of (1).

That is, the real part of $e^{(2 + 3i)x}$ is $e^{2x}\cos 3x$ and the imaginary part is $e^{2x}\sin 3x$.

Now, if $L[e^{(2 + 3i)x}] = 0$, $r_1 = 2 + 3i$ must be a root of $L(e^{rx}) = 0$. Moreover, $r_1 = 2 + 3i$ must be at least a double root of $L(e^{rx}) = 0$ since it is $x(e^{2x}\cos 3x)$, not just $e^{2x}\cos 3x$, which is a solution of (1). Hence, $(r - r_1)^2$ <u>must</u> be a factor of $L(e^{rx})$, where $r_1 = 2 + 3i$. [It is possible that $(r - r_1)^3$, etc., is a factor. For example, $x^2e^{2x}\cos 3x$ might also be a solution of (1) but we are looking for the lowest order equation that has $y = xe^{2x}\cos 3x$ as a solution.]

Since the polynomial factor of $L(e^{rx})$ is the only factor which can equal 0 and since the polynomial has real coefficients, we see that each time $r = r_1$ is a root of $L(e^{rx}) = 0$, so also is $r = \bar{r}_1$.

Thus, $L(e^{rx})$, in this case, must have, <u>at least</u>,

$$(r - r_1)^2(r - \bar{r}_1)^2$$

as a factor.

Now

$$\begin{aligned}(r - r_1)^2(r - \bar{r}_1)^2 &= [(r - r_1)(r - \bar{r}_1)]^2 \\ &= [r^2 - (r_1 + \bar{r}_1)r + |r_1|^2]^2. \end{aligned} \tag{2}$$

We see from (2) that with $r_1 = 2 + 3i$ (so $\bar{r}_1 = 2 - 3i$, $r_1 + \bar{r}_1 = 4$, $|r_1|^2 = 13$)

2.5.7 continued

$$(r^2 - 4r + 13)^2 = r^4 + 16r^2 + 169 - 8r^3 + 26r^2 - 104r$$

$$= r^4 - 8r^3 + 42r^2 - 104r + 169 \qquad (3)$$

is a factor of $L(e^{rx})$.

Therefore, in (3), replacing r by y and exponents by derivatives, we obtain that

$$\frac{d^4y}{dx^4} - 8\frac{d^3y}{dx^3} + 42\frac{d^2y}{dx^2} - 104\frac{dy}{dx} + 169y = 0 \qquad (4)$$

is the lowest order (homogeneous) linear differential equation with constant coefficients which can have $y = xe^{2x} \cos 3x$ as a solution. We humanely spare the reader the details of checking the result, as well as ourselves the embarrassment of uncovering an error in the derivation of (4).

2.5.8

Letting

$$L(y) = \frac{d^6y}{dx^6} - 2\frac{d^3y}{dx^3} + y$$

we see that

$$L(e^{rx}) = e^{rx}(r^6 - 2r^3 + 1)$$

$$= e^{rx}(r^6 - 2r^3 + 1)$$

Thus

$$L(e^{rx}) = 0 \leftrightarrow r^6 - 2r^3 + 1 = 0. \qquad (1)$$

Now

$$r^6 - 2r^3 + 1 = (r^3 - 1)^2 \qquad (2)$$

2.5.8 continued

the factors of $r^3 - 1$ come from the roots of $r^3 = 1$ and these are

$$\left.\begin{aligned} r_1 &= 1 \\ r_2 &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i \\ r_3 &= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i = \bar{r}_2 \end{aligned}\right\} \quad (3)$$

[Of course, (3) is just showing off our knowledge of how to extract roots using complex numbers. A more mundane, but perhaps less frightening a technique in this case would be to factor $r^3 - 1$ as $(r - 1)(r^2 + r + 1)$ and use the quadratic formula on $r^2 + r + 1 = 0$.]

At any rate, from (3), we have that

$$e^x \quad \text{and} \quad e^{(-\frac{1}{2} + i\frac{1}{2}\sqrt{3})x}$$

are solutions of our equation.

Next observe that

$$e^{(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i)x} = e^{-\frac{1}{2}x}\left[\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right],$$

so that $e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$ and $e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$ are a pair of <u>real</u> solutions.

But from (2) we know that each root is a double root so in addition to the solutions

$$y = e^x,\ y = e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x, \text{ and } y = e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$$

we also have the solutions

$$y = xe^x,\ y = xe^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x, \text{ and } y = xe^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x.$$

2.5.8 continued

Thus, the required general solution is

$$y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + (c_5 + c_6x)\sin\frac{\sqrt{3}}{2}x.$$

2.5.9 (Optional)

a. $(xe^{rx})' = xre^{rx} + e^{rx}$

$$= (1 + rx)e^{rx} \qquad (1)$$

$$(xe^{rx})'' = [(1 + rx)e^{rx}]'$$

$$= (1 + rx)re^{rx} + re^{rx}$$

$$= (2r + r^2x)e^{rx} \qquad (2)$$

$$(xe^{rx})''' = [(2r + r^2x)e^{rx}]'$$

$$= (2r + r^2x)re^{rx} + r^2e^{rx}$$

$$= (3r^2 + r^3x)e^{rx} \qquad (3)$$

b. From (1), (2), and (3) it seems a "natural" conjecture that

$$\frac{d^k(xe^{rx})}{dx^k} = (kr^{k-1} + xr^k)e^{rx}. \qquad (4)$$

To establish (4) inductively, since we know from (1), (2), and (3), that it's true for k = 1, 2, 3, we must show that (4) implies

$$\frac{d^{k+1}(xe^{rx})}{dx^{k+1}} = [(k + 1)r^k + xr^{k+1}]e^{rx}. \qquad (5)$$

Well, using (4) and differentiating with respect to x, we obtain

2.5.9 continued

$$\frac{d^{k+1}(xe^{rx})}{dx^k} = (kr^{k-1} + xr^k)re^{rx} + r^k e^{rx}$$

$$= (kr^k + xr^{k+1} + r^k)e^{rx}$$

$$= ([k+1]r^k + xr^{k+1})e^{rx}$$

which agrees with (5). Thus we conclude by induction that

$$\left.\begin{array}{l} \dfrac{d^k(xe^{rx})}{dx^k} = (kr^{k-1} + xr^k)e^{rx} \\ \\ \text{for } k = 0^*,1,2,3,\ldots\ . \end{array}\right\} \qquad (6)^{**}$$

c. $$\frac{d^k(e^{rx})}{dx^k} = r^k e^{rx}. \qquad (7)$$

Hence, from (7)

$$\frac{\partial\left[\dfrac{d^k(e^{rx})}{dx^k}\right]}{\partial r} = \frac{\partial}{\partial r}(r^k e^{rx})$$

$$= r^k xe^{rx} + kr^{k-1}e^{rx}$$

$$= (kr^{k-1} + xr^k)e^{rx}. \qquad (8)$$

Comparing (6) and (8), we see that

$$\frac{\partial L(e^{rx})}{\partial r} = L(xe^{rx})$$

in the <u>special case</u> that $L(y) = \dfrac{d^k y}{dx^k}$.

*We never checked k = 0, but with k = 0, (4) reduces to $xe^{rx} = xe^{rx}$.

**We use (6) even though it looks the same as (4) since (4) was only a conjecture.

2.5.9 continued

The beauty is now that by <u>linearity</u> the general result is established since

$$L\left[\sum_{k=0}^{n} a_k \frac{d^k y}{dx^k}\right] = \sum_{k=0}^{n} L\left(a_k \frac{d^k y}{dx^k}\right)$$

$$= \sum_{k=0}^{n} a_k L\left(\frac{d^k y}{dx^k}\right) \qquad (9)$$

Equation (9) shows that the truth of

$$\frac{\partial L(e^{rx})}{\partial r} = L(xe^{rx})$$

is guaranteed as soon as we know that it is true whenever $L(y)$ is the single term $\frac{d^k y}{dx^k}$, $k = 0,1,2,\ldots$.