SOLUTION OF FINITE ELEMENT EQUILIBRIUM EQUATIONS IN DYNAMIC ANALYSIS

LECTURE 10

56 MINUTES

LECTURE 10 Solution of dynamic response by direct integration

Basic concepts used

Explicit and implicit techniques

Implementation of methods

Detailed discussion of central difference and Newmark methods

Stability and accuracy considerations

Integration errors

Modeling of structural vibration and wave propagation problems

Selection of element and time step sizes

Recommendations on the use of the methods in practice

TEXTBOOK: Sections: 9.1, 9.2.1, 9.2.2, 9.2.3, 9.2.4, 9.2.5, 9.4.1, 9.4.2, 9.4.3, 9.4.4

Examples: 9.1, 9.2, 9.3, 9.4, 9.5, 9.12

DIRECT INTEGRATION SOLUTION OF EQUILIBRIUM EQUATIONS IN DYNAMIC ANALYSIS

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R}$$

- explicit, implicit integration
- computational considerations
- selection of solution time step (Δt)
- some modeling considerations

Equilibrium equations in dynamic analysis

$$\underline{M}\,\ddot{\underline{U}} + \underline{C}\,\dot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (9.1)$$

or

$$\underline{F}_I(t) + \underline{F}_D(t) + \underline{F}_E(t) = \underline{R}(t) \qquad (9.2)$$

Load description

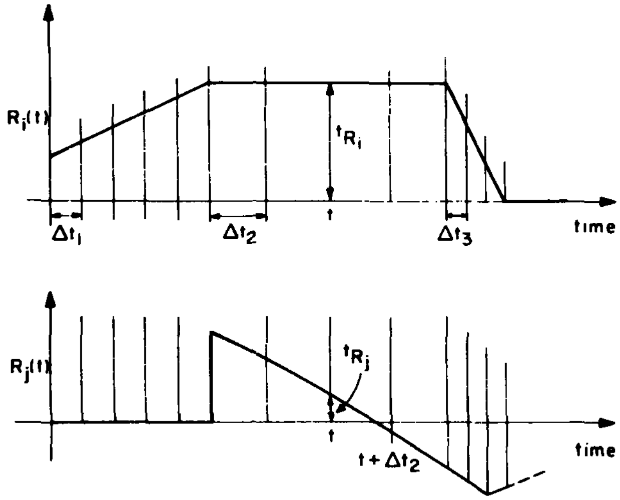

Fig. 1. Evaluation of externally applied nodal point load vector ${}^t\underline{R}$ at time t.

THE CENTRAL DIFFERENCE METHOD (CDM)

$$ {}^t\ddot{\underline{U}} = \frac{1}{\Delta t^2}\{{}^{t-\Delta t}\underline{U} - 2\,{}^t\underline{U} + {}^{t+\Delta t}\underline{U}\} \qquad (9.3)$$

$$ {}^t\dot{\underline{U}} = \frac{1}{2\Delta t}(-{}^{t-\Delta t}\underline{U} + {}^{t+\Delta t}\underline{U}) \qquad (9.4)$$

$$ \underline{M}\,{}^t\ddot{\underline{U}} + \underline{C}\,{}^t\dot{\underline{U}} + \underline{K}\,{}^t\underline{U} = {}^t\underline{R} \qquad (9.5)$$

an explicit integration scheme

Combining (9.3) to (9.5) we obtain

$$\left(\frac{1}{\Delta t^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right){}^{t+\Delta t}\underline{U} = {}^{t}\underline{R} - \left(\underline{K} - \frac{2}{\Delta t^2}\underline{M}\right){}^{t}\underline{U}$$
$$-\left(\frac{1}{\Delta t^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right){}^{t-\Delta t}\underline{U} \qquad (9.6)$$

where we note

$$\underline{K}\,{}^{t}\underline{U} = \left(\sum_m \underline{K}^{(m)}\right){}^{t}\underline{U}$$
$$= \sum_m \left(\underline{K}^{(m)}\,{}^{t}\underline{U}\right) = \sum_m {}^{t}\underline{F}^{(m)}$$

<u>Computational considerations</u>

- to start the solution, use

$${}^{-\Delta t}U_{(i)} = {}^{0}U_{(i)} - \Delta t\,{}^{0}\dot{U}_{(i)} + \frac{\Delta t^2}{2}\,{}^{0}\ddot{U}_{(i)} \qquad (9.7)$$

- in practice, mostly used with lumped mass matrix and low-order elements.

Stability and Accuracy of CDM

- Δt must be smaller than Δt_{cr}

$$\Delta t_{cr} = \frac{T_n}{\pi} \quad ; \quad T_n = \text{smallest natural period in the system}$$

hence method is conditionally stable

- in practice, use for continuum elements,

$$\Delta t \leq \frac{\Delta L}{c} \quad ; \quad c = \sqrt{\frac{E}{\rho}}$$

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for lower-order elements

ΔL = smallest distance between nodes

for high-order elements

ΔL = (smallest distance between nodes)/(rel. stiffness factor)

- method used mainly for wave propagation analysis

- number of operations $\propto$ no. of elements and no. of time steps

THE NEWMARK METHOD

$$ {}^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + [(1-\delta)\,{}^{t}\ddot{\underline{U}} + \delta\,{}^{t+\Delta t}\ddot{\underline{U}}]\,\Delta t \qquad (9.27) $$

$$ {}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + {}^{t}\dot{\underline{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\underline{U}} + \alpha\,{}^{t+\Delta t}\ddot{\underline{U}}]\,\Delta t^2 \qquad (9.28) $$

$$ \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}} + \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}} + \underline{K}\,{}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\underline{R} \qquad (9.29) $$

an implicit integration scheme solution is obtained using

$$ \hat{\underline{K}}\,{}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\hat{\underline{R}} $$

- **In practice, we use mostly**

  $\alpha = \frac{1}{4}\ ,\ \delta = \frac{1}{2}$

  which is the

  constant-average-acceleration method (Newmark's method)

- **method is unconditionally stable**
- **method is used primarily for analysis of structural dynamics problems**
- **number of operations**

  $\doteq \frac{1}{2} n m^2 + 2 n m t$

Accuracy considerations

- time step Δt is chosen based on accuracy considerations only
- Consider the equations

$$\underline{M}\,\ddot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R}$$

and

$$\underline{U} = \sum_{i=1}^{n} \underline{\phi}_i\, x_i(t)$$

where

$$\underline{K}\,\underline{\phi}_i = \omega_i^2\,\underline{M}\,\underline{\phi}_i$$

Using

$$\underline{\Phi}^T\,\underline{K}\,\underline{\Phi} = \underline{\Omega}^2 \quad ; \quad \underline{\Phi}^T\,\underline{M}\,\underline{\Phi} = \underline{I}$$

where

$$\underline{\Phi} = [\underline{\phi}_1, \ldots, \underline{\phi}_n] \quad ; \quad \underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix}$$

we obtain n equations from which to solve for $x_i(t)$ (see Lecture 11)

$$\ddot{x}_i + \omega_i^2\, x_i = \underline{\phi}_i^T\,\underline{R} \qquad i = 1,\ldots,n$$

Hence, the direct step-by-step solution of

$$\underline{M}\,\ddot{\underline{U}} + \underline{K}\,\underline{U} = \underline{R}$$

corresponds to the direct step-by-step solution of

$$\ddot{x}_i + \omega_i^2\, x_i = \underline{\phi}_i^T\,\underline{R} \qquad i = 1,\ldots,n$$

with

$$\underline{U} = \sum_{i=1}^{n} \underline{\phi}_i\, x_i$$

Therefore, to study the accuracy of the Newmark method, we can study the solution of the single degree of freedom equation

$$\ddot{x} + \omega^2 x = r$$

Consider the case

$$\ddot{x} + \omega^2 x = 0$$

$${}^0x = 1.0 \quad ; \quad {}^0\dot{x} = 0 \quad ; \quad {}^0\ddot{x} = -\omega^2$$

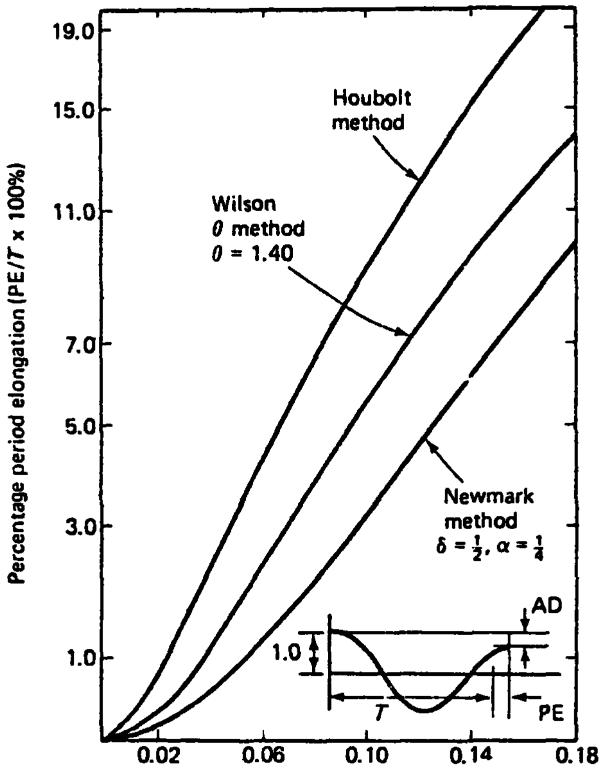


Fig. 9.8 (a) Percentage period elongations and amplitude decays.

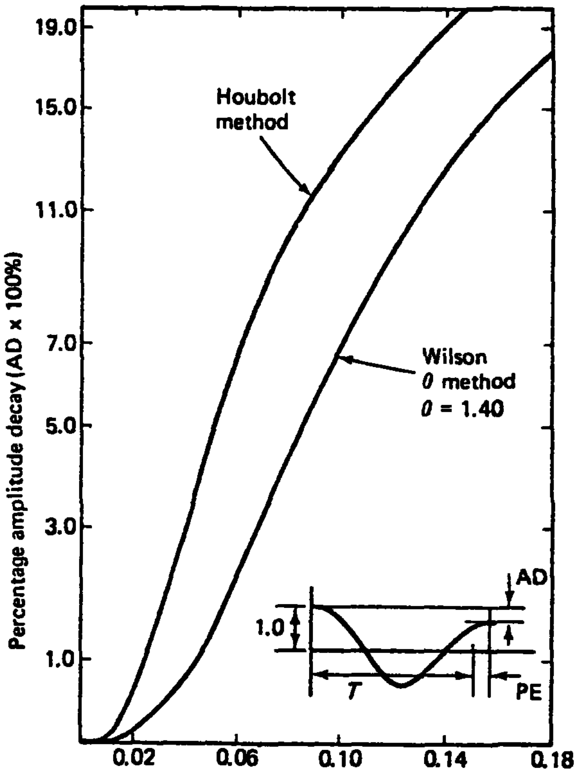


Fig. 9.8 (b) Percentage period elongations and amplitude decays.

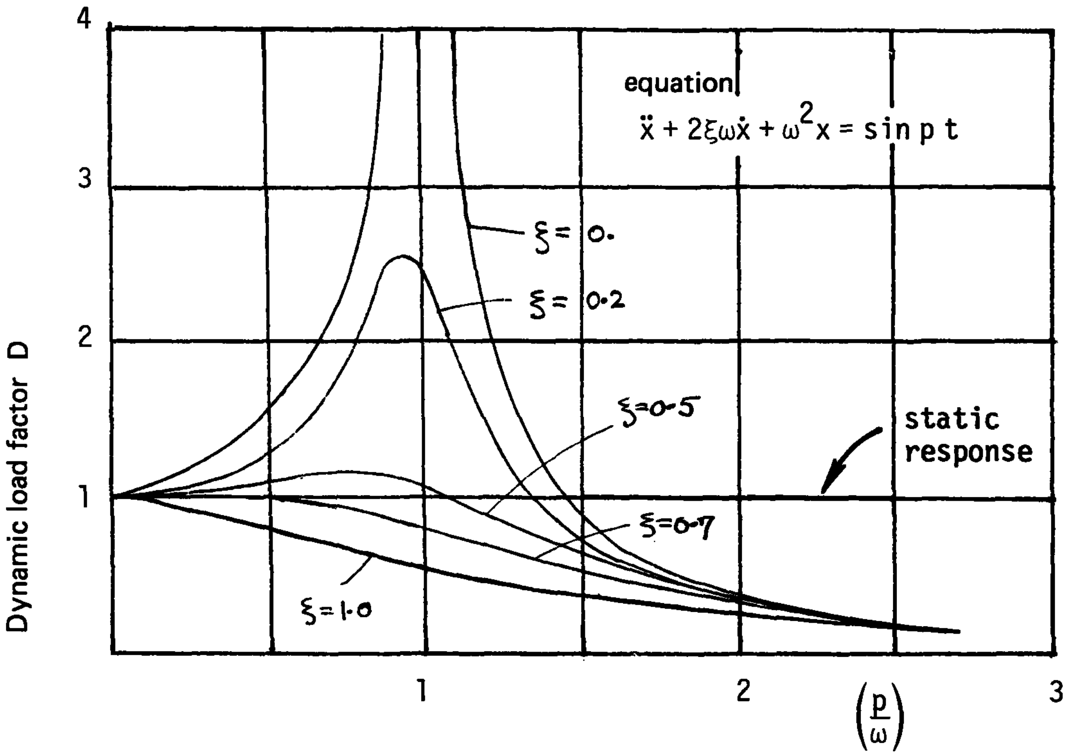
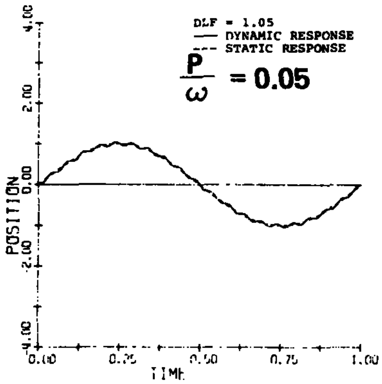
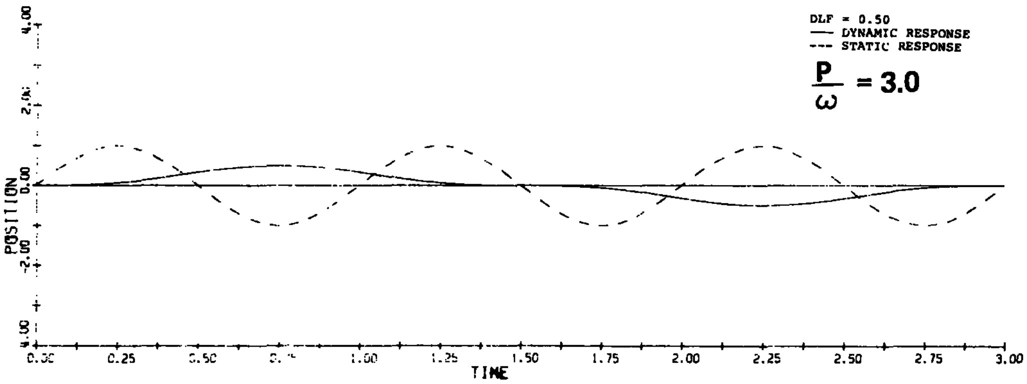


Fig. 9.4. The dynamic load factor



Response of a single degree of freedom system.



Response of a single degree of freedom system.

Modeling of a structural vibration problem

1) Identify the frequencies contained in the loading, using a Fourier analysis if necessary.

2) Choose a finite element mesh that accurately represents all frequencies up to about four times the highest frequency ω_u contained in the loading.

3) Perform the direct integration analysis. The time step Δt for this solution should equal about $\frac{1}{20} T_u$, where $T_u = 2\pi/\omega_u$, or be smaller for stability reasons.

Modeling of a wave propagation problem

If we assume that the wave length is L_w, the total time for the wave to travel past a point is

$$t_w = \frac{L_w}{c} \qquad (9.100)$$

where c is the wave speed. Assuming that n time steps are necessary to represent the wave, we use

$$\Delta t = \frac{t_w}{n} \qquad (9.101)$$

and the "effective length" of a finite element should be

$$L_e = c\ \Delta t \qquad (9.102)$$

SUMMARY OF STEP-BY-STEP INTEGRATIONS

— INITIAL CALCULATIONS —

1. Form linear stiffness matrix $\underline{K}$, mass matrix $\underline{M}$ and damping matrix $\underline{C}$, whichever applicable;

Calculate the following constants:

Newmark method: $\delta \geq 0.50$, $\alpha \geq 0.25(0.5+\delta)^2$

$a_0 = 1/(\alpha\Delta t^2)$	$a_1 = \delta/(\alpha\Delta t)$	$a_2 = 1/(\alpha\Delta t)$	$a_3 = 1/(2\alpha)-1$
$a_4 = \delta/\alpha - 1$	$a_5 = \Delta t(\delta/\alpha - 2)/2$	$a_6 = a_0$	$a_7 = -a_2$
$a_8 = -a_3$	$a_9 = \Delta t(1-\delta)$	$a_{10} = \delta\Delta t$	

Central difference method:

$a_0 = 1/\Delta t^2$	$a_1 = 1/2\Delta t$	$a_2 = 2a_0$	$a_3 = 1/a_2$

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2. Initialize ${}^0\underline{U}$, ${}^0\dot{\underline{U}}$, ${}^0\ddot{\underline{U}}$;

For central difference method only, calculate ${}^{\Delta t}\underline{U}$ from initial conditions:

$${}^{\Delta t}\underline{U} = {}^0\underline{U} + \Delta t\ {}^0\dot{\underline{U}} + a_3\ {}^0\ddot{\underline{U}}$$

3. Form effective linear coefficient matrix;

in implicit time integration:

$$\hat{\underline{K}} = \underline{K} + a_0\underline{M} + a_1\underline{C}$$

in explicit time integration:

$$\hat{\underline{M}} = a_0\underline{M} + a_1\underline{C}$$

4. In dynamic analysis using implicit time integration triangularize $\hat{\underline{K}}$.

--- FOR EACH STEP ---

(i) Form effective load vector;

in implicit time integration:

$$^{t+\Delta t}\hat{\underline{R}} = {}^{t+\Delta t}\underline{R} + \underline{M}(a_0\, {}^{t}\underline{U} + a_2\, {}^{t}\dot{\underline{U}} + a_3\, {}^{t}\ddot{\underline{U}})$$

$$+ \underline{C}(a_1\, {}^{t}\underline{U} + a_4\, {}^{t}\dot{\underline{U}} + a_5\, {}^{t}\ddot{\underline{U}})$$

in explicit time integration:

$$^{t}\hat{\underline{R}} = {}^{t}\underline{R} + a_2\underline{M}({}^{t}\underline{U} - {}^{t-\Delta t}\underline{U}) + \hat{\underline{M}}\, {}^{t-\Delta t}\underline{U} - {}^{t}\underline{F}$$

(ii) Solve for displacement increments;

in implicit time integration:

$$\hat{\underline{K}}\, {}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\hat{\underline{R}} \; ; \; \underline{U} = {}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}$$

in explicit time integration:

$$\hat{\underline{M}}\, {}^{t+\Delta t}\underline{U} = {}^{t}\hat{\underline{R}}$$

Newmark Method:

$$^{t+\Delta t}\ddot{\underline{U}} = a_6\ \underline{U} + a_7\ ^{t}\dot{\underline{U}} + a_8\ ^{t}\ddot{\underline{U}}$$

$$^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + a_9\ ^{t}\ddot{\underline{U}} + a_{10}\ ^{t+\Delta t}\ddot{\underline{U}}$$

$$^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + \underline{U}$$

Central Difference Method:

$$^{t}\dot{\underline{U}} = a_1(^{t+\Delta t}\underline{U} - {}^{t-\Delta t}\underline{U})$$

$$^{t}\ddot{\underline{U}} = a_0(^{t+\Delta t}\underline{U} - 2\,^{t}\underline{U} + {}^{t-\Delta t}\underline{U})$$