FORMULATION AND CALCULATION OF ISOPARAMETRIC MODELS

LECTURE 6

57 MINUTES

LECTURE 6

Formulation and calculation of isoparametric continuum elements

Truss, plane-stress, plane-strain, axisymmetric and three-dimensional elements

Variable-number-nodes elements, curved elements

Derivation of interpolations, displacement and strain interpolation matrices, the Jacobian transformation

Various examples; shifting of internal nodes to achieve stress singularities for fracture mechanics analysis

TEXTBOOK: Sections: 5.1, 5.2, 5.3.1, 5.3.3, 5.5.1

Examples: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17

FORMULATION AND CALCULATION OF ISO-PARAMETRIC FINITE ELEMENTS

- We considered earlier (lecture 4) generalized coordinate finite element models
- We now want to discuss a more general approach to deriving the required interpolation matrices and element matrices $\Longrightarrow$ isoparametric elements

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Isoparametric Elements
Basic Concept: (Continuum Elements)

Interpolate Geometry

$$x = \sum_{i=1}^{N} h_i x_i \; ; \quad y = \sum_{i=1}^{N} h_i y_i \; ; \quad z = \sum_{i=1}^{N} h_i z_i$$

Interpolate Displacements

$$u = \sum_{i=1}^{N} h_i u_i \quad v = \sum_{i=1}^{N} h_i v_i \quad w = \sum_{i=1}^{N} h_i w_i$$

N = number of nodes

1/D Element	Truss	Continuum Elements
2/D Elements	Plane stress Plane strain Axisymmetric Analysis	
3/D Elements	Three-dimensional Thick Shell	

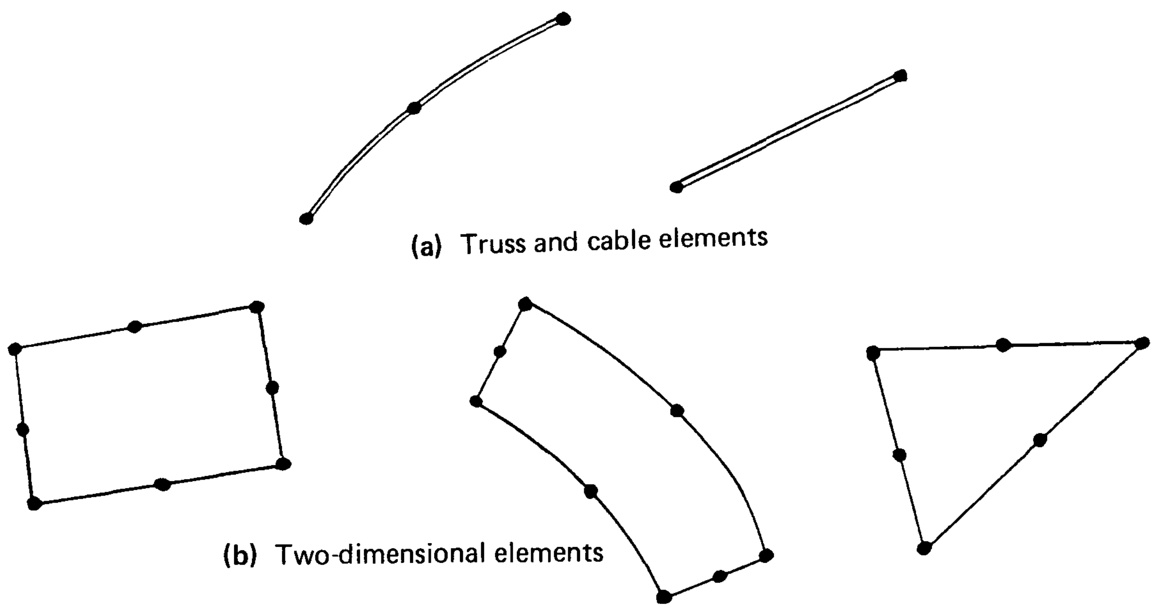
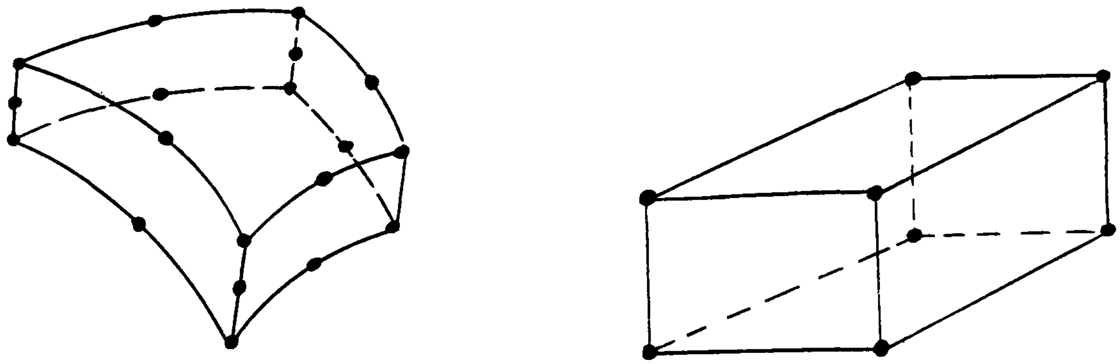


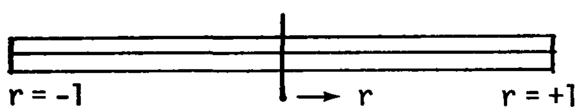
Fig. 5.2. Some typical continuum elements



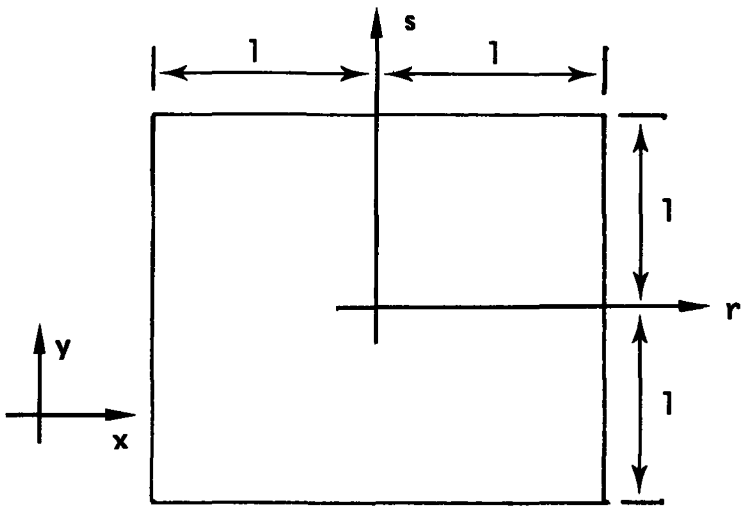
(c) Three-dimensional elements

Fig. 5.2. Some typical continuum elements

Consider special geometries first:



Truss, 2 units long

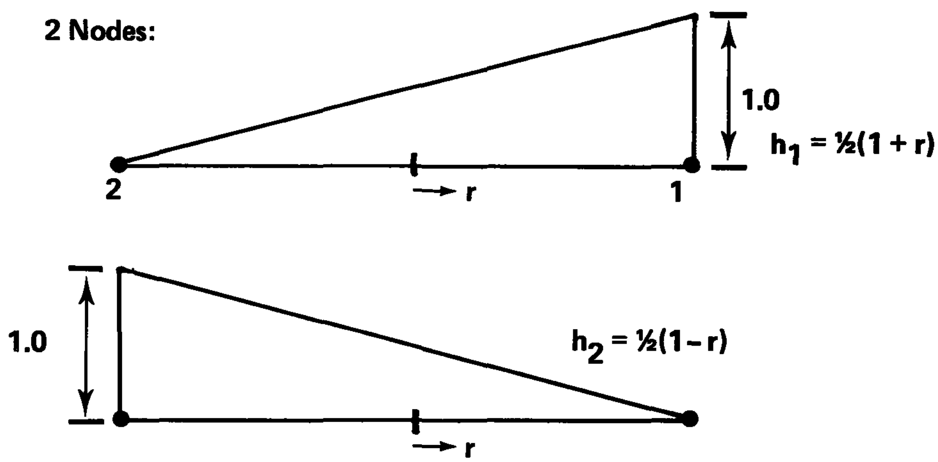


2/D element, 2x2 units

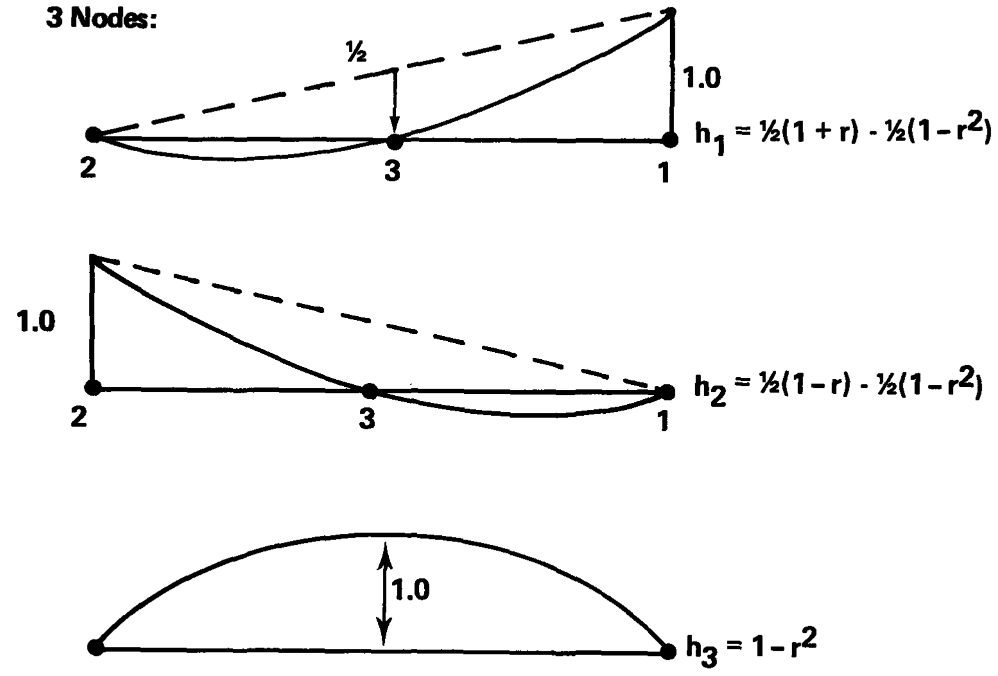
Similarly 3/D element 2x2x2 units
(r - s - t axes)

1 - D Element

2 Nodes:



3 Nodes:



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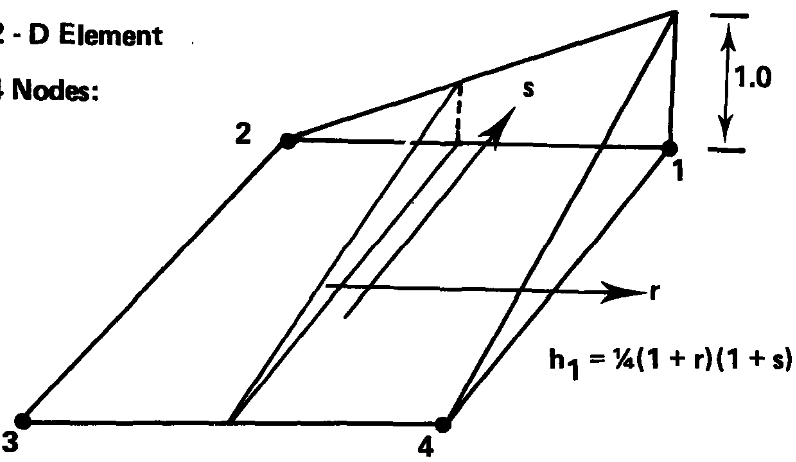
2 - D Element

4 Nodes:



Similarly

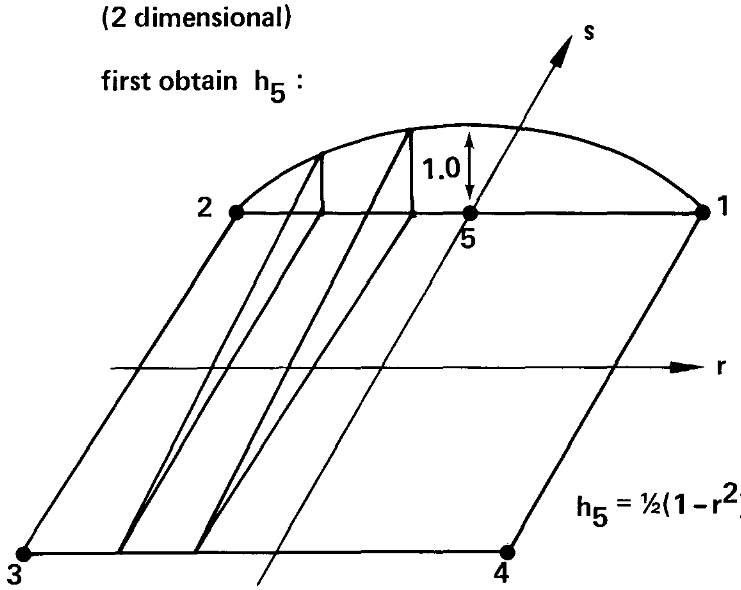
$h_2 = \frac{1}{4}(1 - r)(1 + s)$

$h_3 = \frac{1}{4}(1 - r)(1 - s)$

$h_4 = \frac{1}{4}(1 + r)(1 - s)$

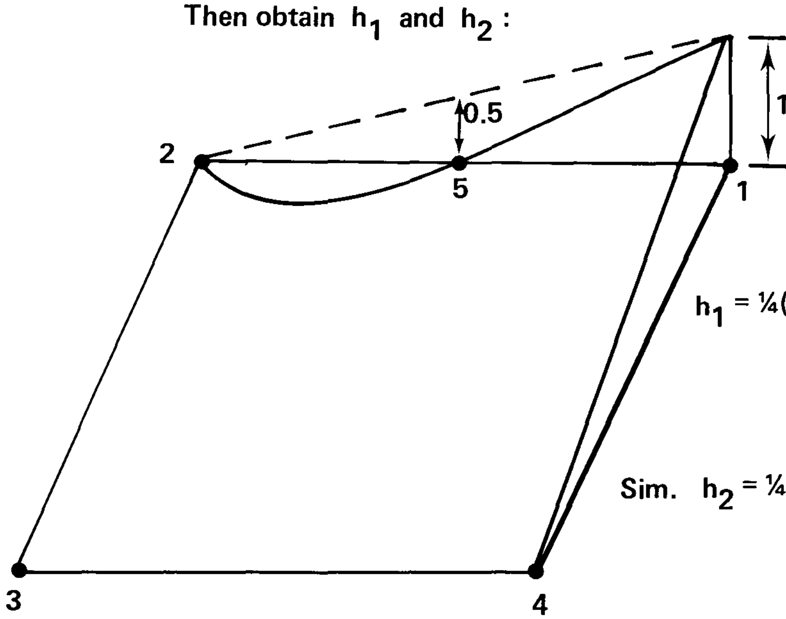
Construction of 5 node element
(2 dimensional)

first obtain h_5 :



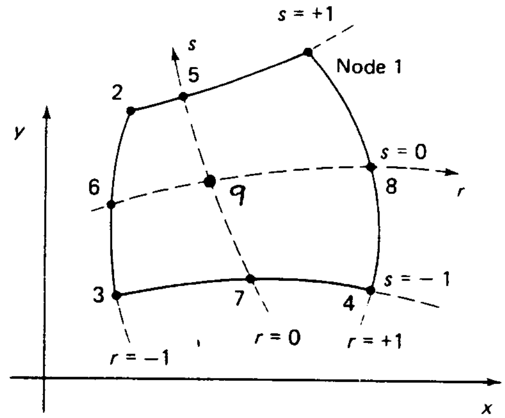
$h_5 = \frac{1}{2}(1 - r^2)(1 + s)$

Then obtain h_1 and h_2 :



$h_1 = \frac{1}{4}(1 + r)(1 + s) - \frac{1}{2}h_5$

Sim. $h_2 = \frac{1}{4}(1 - r)(1 + s) - \frac{1}{2}h_5$



(a) Four to 9 variable-number-nodes two-dimensional element

Fig. 5.5. Interpolation functions of four to nine variable-number-nodes two-dimensional element.

		Include only if node i is defined				
		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions

Fig. 5.5. Interpolation functions of four to nine variable-number-nodes two-dimensional element.

Having obtained the h_i we can construct the matrices $\underline{H}$ and $\underline{B}$:

- The elements of $\underline{H}$ are the h_i (or zero)
- The elements of $\underline{B}$ are the derivatives of the h_i (or zero),

Because for the 2x2x2 elements we can use

$$x \equiv r$$
$$y \equiv s$$
$$z \equiv t$$

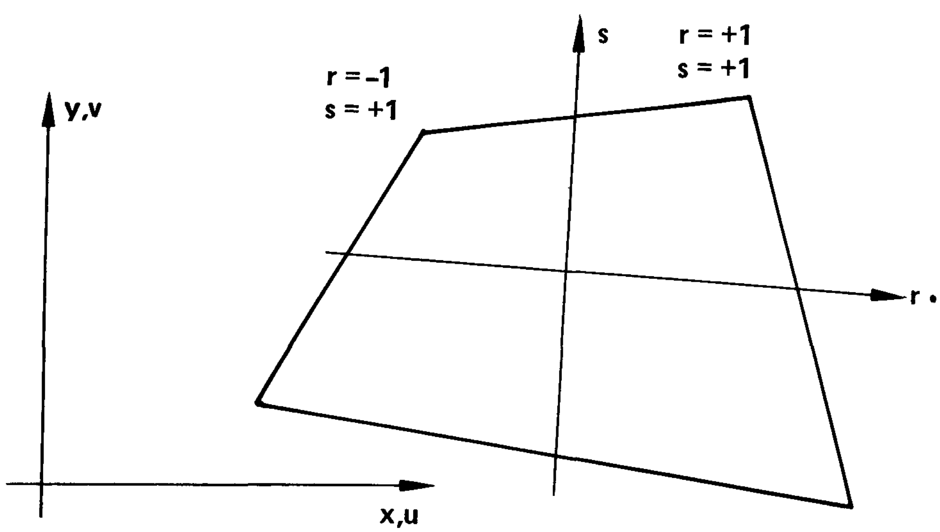
EXAMPLE 4 node 2 dim. element

$$\begin{bmatrix} u(r,s) \\ v(r,s) \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc|cc|cc} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{array}\right]}_{\underline{H}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ v_4 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{ss} \\ \gamma_{rs} \end{bmatrix} = \underbrace{\left[\begin{array}{cc|c|cc} \frac{\partial h_1}{\partial r} & 0 & & \frac{\partial h_4}{\partial r} & 0 \\ 0 & \frac{\partial h_1}{\partial s} & \cdots & 0 & \frac{\partial h_4}{\partial s} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_1}{\partial r} & & \frac{\partial h_4}{\partial s} & \frac{\partial h_4}{\partial r} \end{array}\right]}_{\underline{B}} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ v_4 \end{bmatrix}$$

We note again $r \equiv x$
$s \equiv y$

GENERAL ELEMENTS



Displacement and geometry interpolation as before, but

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

Aside:
cannot use

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \frac{\partial r}{\partial x} + \ldots$$

or

$$\underline{\frac{\partial}{\partial r}} = \underline{J} \, \underline{\frac{\partial}{\partial x}} \quad \text{(in general)}$$

$$\underline{\frac{\partial}{\partial x}} = \underline{J}^{-1} \, \underline{\frac{\partial}{\partial r}} \qquad (5.25)$$

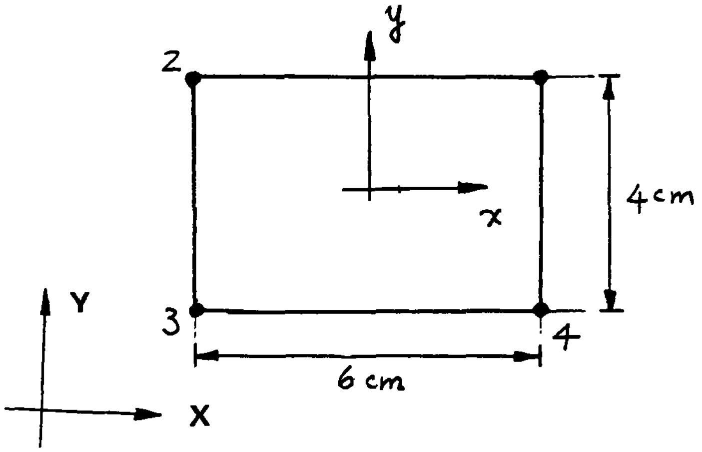
Using (5.25) we can find the matrix $\underline{B}$ of general elements

The $\underline{H}$ and $\underline{B}$ matrices are a function of r , s , t ; for the integration thus use

$$dv = \det \underline{J} \; dr \, ds \, dt$$

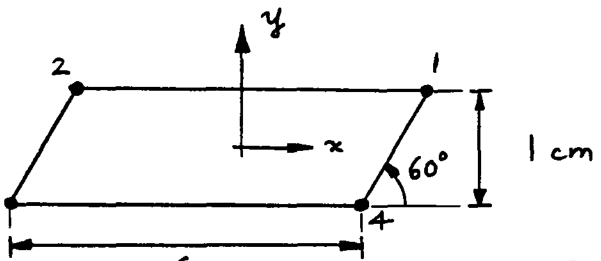
Fig. 5.9. Some two-dimensional elements

Element 1



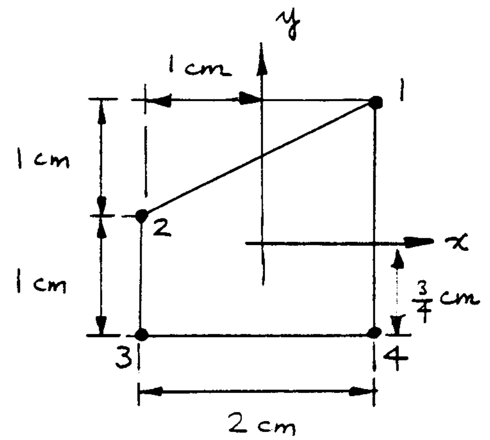
$$\underline{J} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Element 2



$$\underline{J} = \begin{bmatrix} 3 & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}$$

Element 3



$$\underline{J} = \begin{bmatrix} 4 & (1+s) \\ 0 & (3+r) \end{bmatrix}$$

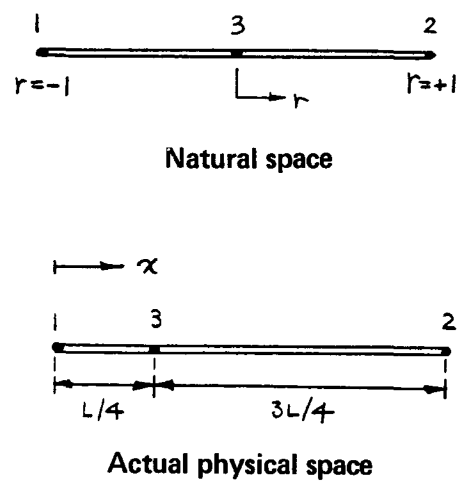


Fig. 5.23. Quarter-point one-dimensional element.

Here we have

$$x = \sum_{i=1}^{3} h_i x_i \Rightarrow x = \frac{L}{4}(1+r)^2$$

hence

$$\underline{J} = \left[\frac{L}{2} + \frac{r}{2}L\right]$$

and

$$\underline{B} = \frac{1}{\frac{L}{2}+\frac{r}{2}L} \left[h_{1,r} \quad h_{2,r} \quad h_{3,r}\right]$$

or

$$\underline{B} = \frac{1}{\frac{L}{2}+\frac{r}{2}L} \left[(-\tfrac{1}{2}+r) \quad (\tfrac{1}{2}+r) \quad -2r\right]$$

Since

$$r = 2\sqrt{\frac{x}{L}} - 1$$

$$\underline{B} = \Big[\Big(\frac{2}{L} - \frac{3}{2\sqrt{L}} \frac{1}{\sqrt{x}}\Big) \quad \Big(\frac{2}{L} - \frac{1}{2\sqrt{L}} \frac{1}{\sqrt{x}}\Big)$$

$$\Big(\frac{2}{\sqrt{L}\,\sqrt{x}} - \frac{4}{L}\Big)\Big]$$

We note

$\frac{1}{\sqrt{x}}$ singularity at $X = 0$!

Numerical Integration

Gauss Integration
Newton-Cotes Formulas

$$\underline{K} = \sum_{i,j,k} \alpha_{ijk} \underline{F}_{ijk}$$

$$\underline{F} = \underline{B}^T \underline{C} \underline{B} \det \underline{J}$$