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SOLUTIONS TO CHAPTER 3

3.1 TEMPORAL EVOLUATION OF WORLD GOVERNED BY LAWS OF MAXWELL, LORENTZ, AND NEWTON

3.1.1 (a) Replace z by $z - ct$. Thus

$$\mathbf{E} = E_o \mathbf{i_x} e^{-(z-ct)^2/2a^2}; \quad \mathbf{H} = \sqrt{\frac{\epsilon_o}{\mu_o}} E_o \mathbf{i_y} e^{-(z-ct)^2/2a^2} \tag{1}$$

(b) Because $\partial(\ \ )/\partial x = \partial(\ \ )/\partial y = 0$ and there are only single components of each field, Maxwell's equations reduce to

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_o}\frac{\partial E_x}{\partial z}; \quad \frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_o}\frac{\partial H_y}{\partial z} \tag{2}$$

Note that we could pick these expressions out of the six components of the laws of Faraday and Ampére by first writing the left hand sides of 3.1.1-2. Thus, these are respectively the y and x components of these laws. In Cartesian coordinates, the divergence equations are automatically satisfied by any vector that only depends on a coordinate perpendicular to its direction. Substitution of (1) into (2a) and into (2b) gives

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{3}$$

which is the velocity of light, in agreement with (3.1.16).

(c) For an observer having the location $z = ct+$ constant, whose position increases linearly with time at the rate c m/s and who therefore has the constant velocity c, $z - ct =$ constant. Thus, the fields given by (1) are constant.

3.1.2 With the given substitution in (3.1.1-4), (with $\mathbf{J} = 0$ and $\rho = 0$)

$$-\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\epsilon_o}\nabla \times \mathbf{H} \tag{1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_o}\nabla \times \mathbf{E} \tag{2}$$

$$0 = \nabla \cdot \mu_o \mathbf{H} \tag{3}$$

$$0 = -\nabla \cdot \epsilon_o \mathbf{E} \tag{4}$$

Although reordered, the expressions are the same as the original relations.

3.1.3 Note that the direction of wave propagation is obtained by crossing **E** into **H**. Because it would reverse the direction of this cross product, a good guess is to reverse the sign of one or the other of the fields. In that case, the steps followed in Prob. 3.1.1 lead to the requirement that $c = -1/\sqrt{\mu_o\epsilon_o}$. We define c as being positive and so write the solutions with $z-ct$ replaced by $z-(-c)t = z+ct$. Following the same arguments as in part (c) of Prob. 3.1.1, this solution is therefore traveling in the $-z$ direction.

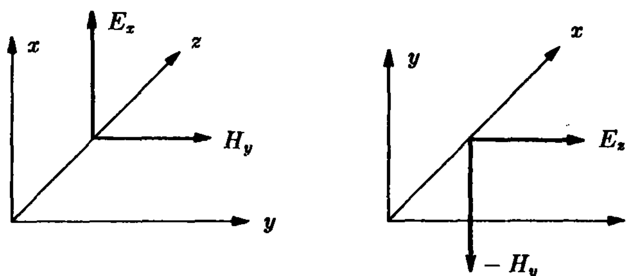

Figure S3.1.4

3.1.4 The role played by z is now taken by x, as shown in Fig. S3.1.4. With the understanding that the z dependence is now replaced by the given x dependence, the magnetic and electric fields are written so that they have the same ratio as in (1) of Prob. 3.1.1. Further, in order to preserve the vector relation between **E**, **H** and the direction of propagation, the sign of **H** is reversed. Thus,

$$\mathbf{E} = E_o\mathbf{i}_z \cos\beta(x - ct); \qquad \mathbf{H} = -\sqrt{\frac{\epsilon_o}{\mu_o}} E_o\mathbf{i}_y \cos\beta(x - ct) \tag{1}$$

3.2 QUASISTATIC LAWS

3.2.1 (a) These fields are transverse to the coordinate, x, upon which they depend. Therefore, the divergence conditions are automatically satisfied. From the direction of the vectors, we know that the x and y components respectively of the laws of Ampére and Faraday will apply.

$$-\frac{\partial H_y}{\partial z} = \frac{\partial \epsilon_o E_x}{\partial t} \tag{1}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial \mu_o H_y}{\partial t} \tag{2}$$

The other four components of these equations are automatically satisfied because $\partial(\ )/\partial y = \partial(\ )/\partial z = 0$. Substitution of (a) and (b) then gives

$$\beta = \omega\sqrt{\mu_o\epsilon_o} \equiv \frac{\omega}{c} \tag{3}$$

in each case.

(b) The appropriate identities are

$$\cos\beta z\cos\omega t = \frac{1}{2}\Big[\cos\beta\big(z-\frac{\omega}{\beta}t\big)+\cos\beta\big(z+\frac{\omega}{\beta}t\big)\Big] \tag{4}$$

$$\sin\beta z\sin\omega t = \frac{1}{2}\Big[\cos\beta\big(z-\frac{\omega}{\beta}t\big)-\cos\beta\big(z+\frac{\omega}{\beta}t\big)\Big] \tag{5}$$

Thus, in view of (3), the fields indeed take the form of the sum of waves traveling in the $+z$ and $-z$ directions with the speed c.

(c) In view of (a), this condition can be written as

$$\beta l = \omega\sqrt{\mu_o\epsilon_o}l = \omega l/c \ll 1 \tag{6}$$

Thus, the condition is equivalent to having the electromagnetic delay time $\tau_{em} \equiv l/c$ short compared to the time $1/\omega$ required for $1/2\pi$ of a cycle.

(d) In the limit of (c), $\cos\beta z \to 1$ and $\sin\beta z \to \beta z$ and (a) and (b) become the given fields.

(e) The electric field of (c) is irrotational and hence satisfies (3.2.1a) but not (3.2.1b) while the magnetic field has curl and indeed satisfies (3.2.2a) but not (3.2.2b). Therefore, in the limit of having the frequency low enough to satisfy (6), the system is EQS.

3.2.2 (a) See part (a) of solution to Prob. 3.2.1.

(b) The appropriate identities are

$$\sin(\beta z)\sin(\omega t) = \frac{1}{2}\Big[\cos\beta\big(z-\frac{\omega}{\beta}t\big)+\cos\beta\big(z+\frac{\omega}{\beta}t\big)\Big] \tag{1}$$

$$\cos(\beta z)\cos(\omega t) = \frac{1}{2}\Big[\cos\beta\big(z-\frac{\omega}{\beta}t\big)-\cos\beta\big(z+\frac{\omega}{\beta}t\big)\Big] \tag{2}$$

Thus, because $\omega/\beta = c$, the fields indeed take the form of the sum of waves traveling in the $+z$ and $-z$ directions with the speed c.

(c) See (c) of solution to Prob. 3.2.1.

(d) In the limit where $|\beta l| \ll 1$, the given fields become

$$\mathbf{E} \simeq \omega\mu_o H_o z\sin\omega t\mathbf{i_x} \tag{3}$$

$$\mathbf{H} \simeq H_o\cos\omega t\mathbf{i_y} \tag{4}$$

Thus, the magnetic field is uniform while the electric field varies linearly between the source and the "short" at $z=0$, where it is zero.

(e) The magnetic field of (4) is irrotational and hence satisfies (3.2.2b) with $\mathbf{J}=0$ but not (3.2.2a). The electric field of (3) does have a curl and hence does not satisfy (3.2.1a) but does satisfy (3.2.1b). Thus, the system is magnetoquasistatic.

3.3 CONDITIONS FOR FIELDS TO BE QUASISTATIC

3.3.1 (a) Except that it is in the x direction rather than the z direction, the quasistatic electric field between the plates is, as in Example 3.3.1, uniform. To satisfy the requirement of (a), this field is

$$\mathbf{E} = [v(t)/d]\mathbf{i}_x \tag{1}$$

The surface charge density on the plates follows from Gauss' integral law applied to the plates, much as in (3.3.7).

$$\sigma_s = \begin{cases} -\epsilon_o E_x(x = d) = -\epsilon_o v/a; & x = d \\ \epsilon_o E_x(x = 0) = \epsilon_o v/d; & x = 0 \end{cases} \tag{2}$$

Thus, the quasistatic surface charge density on the interior surfaces of each plate is uniform.

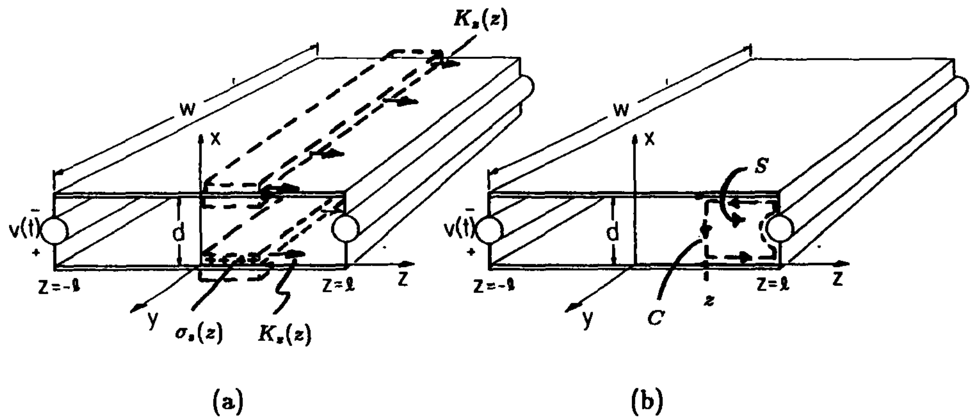

Figure S3.3.1

(b) The integral form of charge conservation is applied to the lower and upper electrodes using the volume shown in Fig. S3.3.1a. Thus, using symmetry to argue that $K_z = 0$ at $z = 0$, for the lower plate

$$w[K_z(z) - K_z(0)] + \frac{\partial \sigma_s z w}{\partial t} = 0 \Rightarrow K_z(z) = -\frac{z\epsilon_o}{d}\frac{dv}{dt} \tag{3}$$

and we conclude that the surface current density increases linearly from the center toward the edges. At any location z, it is that current required to change the charge on the fraction of "capacitor" at a lesser value of z.

(c) The magnetic field is found using Ampére's integral law, (3.3.9), with the surface $da = \mathbf{i}_x da$ having edges at $z = 0$ and $z = z$. By symmetry, $H_y = 0$ at $z = 0$, so

$$w[-H_y(z) + H_y(0)] = w\int_0^z \epsilon_o E_x dz = \frac{\epsilon_o w}{d} z \frac{dv}{dt} \Rightarrow H_y(z) = \frac{\epsilon_o z}{d}\frac{dv}{dt} \tag{4}$$

Note that, with this field and the surface current density of (3), Ampère's continuity condition, 1.4.16, is satisfied on the upper and lower plates. We could just as well think of the magnetic field as being induced by the surface current of (3) as by the displacement current of (3.3.9).

(d) To determine the correction electric field, use Faraday's integral law with the surface and contour shown in Fig. S3.3.1b, assuming that $\mathbf{E}$ is independent of x.

$$d[E_x(0) - E_x(z)] = -\mu_o d\frac{\partial}{\partial t}\int_z^l H_y dz = \frac{-\mu_o\epsilon_o d}{2d}(l^2 - z^2)\frac{d^2v}{dt^2} \tag{5}$$

Because of (a), it follows that the corrected field is

$$E_x(z) = \frac{v}{d} + \frac{\mu_o\epsilon_o}{2d}(l^2 - z^2)\frac{d^2v}{dt^2} \tag{6}$$

(e) With the second term in (6) called the "correction field," it follows that for the given sinusoidally varying voltage, the ratio of the correction field to the quasistatic field at at most

$$\frac{E_{\text{correction}}}{v/d} = \frac{\mu_o\epsilon_o l^2}{2}\frac{1}{|v|}\left|\frac{d^2v}{dt^2}\right| = \frac{\mu_o\epsilon_o l^2\omega^2}{2} \tag{7}$$

Thus, because $c = 1/\sqrt{\mu_o\epsilon_o}$, the error is negligible if

$$\frac{1}{2}\left[\frac{l}{c}\omega\right] \ll 1 \tag{8}$$

3.3.2 (a) With the understanding that the magnetic field outside the structure is zero, Ampèr'es continuity condition, (1.4.16), requires that

$$0 - H_y = K_y = K \qquad \text{top plate}$$

$$H_y - 0 = K_y = -K \qquad \text{bottom plate} \tag{1}$$

where it is recognized that if the current is essentially steady, the surface current densities must be of equal magnitude $K(t)$ and opposite directions in the top and bottom plates. These boundary conditions also require that

$$\mathbf{H} = -\mathbf{i}_y K(t) \tag{2}$$

at the surface current density sources at the left and right as well. Thus, provided $K(t)$ is essentially steady, (2) is taken as holding everywhere between the plates. Note that this uniform distribution of field not only satisfies the boundary conditions, but also has no curl and hence satisfies the steady form of Ampère's law, (3.2.2b), in the region between the plates where $J = 0$.

(b) The integral form of Faraday's law is used to compute the electric field caused by the time variation of $K(t)$.

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_S \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{3}$$

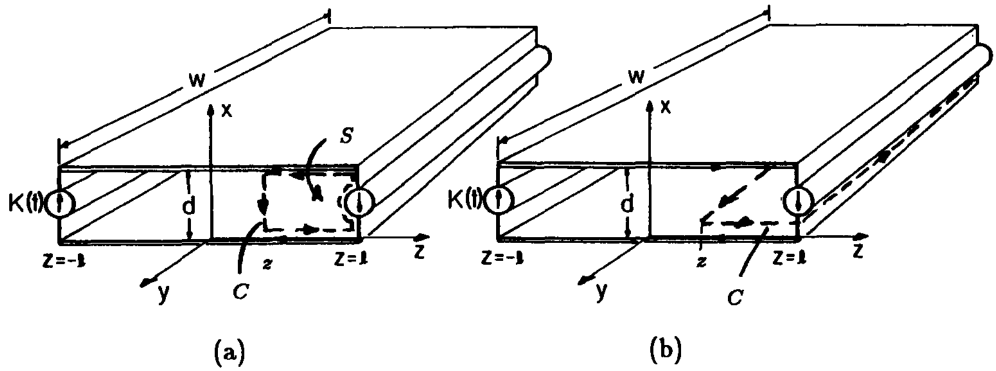

Figure S3.3.2

So that it links the magnetic flux, the surface is chosen to be in the $x-z$ plane, as shown in Fig. S3.3.2a. The upper and lower edges are adjacent to the perfect conductor and therefore do not contribute to the line integral of $\mathbf{E}$. The left edge is at $z = 0$ while the right edge is at some arbitrary position z. Thus, with the assumption that E_y is independent of x,

$$d[E_x(z) - E_x(0)] = -\mu_o z d \frac{\partial H_y}{\partial t} = \mu_o z d \frac{dK}{dt} \tag{4}$$

Thus the electric field is $E_x(0)$ plus an odd function of z. Symmetry requires that $E_x(0) = 0$ so that the desired electric field induced through Faraday's law by the time varying magnetic field is

$$E_x(z) = \mu_o z \frac{dK}{dt} \tag{5}$$

Note that the fields given by (2) and (5) satisfy the MQS field laws in the region between the plates.

(c) To compute the correction to $\mathbf{H}$ that results because of the displacement current, we use the integral form of Ampère's law with the surface shown in Fig. S3.3.2. The right edge is at the surface of the current source, where Ampère's continuity condition requires that $H_y(l) = -K(t)$, and the left edge is at the arbitrary location z. Thus,

$$w[-H_y(l) + H_y(z)] = w\epsilon_o \frac{\partial}{\partial t} \int_z^l E_x dz \tag{6}$$

and so, from this first order correction, we have found that the field is

$$H_y = -K(t) + \frac{w\epsilon_o\mu_o}{w}\frac{(l^2 - z^2)}{2}\frac{d^2K}{dt^2} \tag{7}$$

(d) The second term in (7) is the correction field, so, at worst where $z = 0$,

$$\frac{|H_{\text{corrected}}|}{|K|} = \frac{\epsilon_o\mu_o l^2}{2}\frac{1}{|K|}\Big|\frac{d^2K}{dt^2}\Big| \tag{8}$$

and, for the sinusoidal excitation, we have a negligible correction if

$$\frac{\epsilon_o\mu_o l^2\omega^2}{2} = \frac{1}{2}\Big(\frac{l}{c}\omega\Big)^2 \tag{9}$$

Thus, the correction can be ignored (and hence the MQS approximation is justified) if the electromagnetic transit time l/c is short compared to the typical time $1/\omega$.

3.4 QUASISTATIC SYSTEMS

3.4.1 (a) Using Ampère's integral law, (3.4.2), with the contour and surface shown in Fig. 3.4.2c gives

$$2\pi r H_\phi = 2\pi b K_o(t) \Rightarrow H_\phi = \frac{b}{r}K_o(t) \tag{1}$$

(b) For essentially steady currents, the net current in the z direction through the inner distributed surface current source must equal that radially outward at any radius r in the upper surface, must equal that in the $-z$ direction in the outer wall and must equal that in the $-r$ direction at any radius r in the lower wall. Thus,

$$\begin{aligned} 2\pi b K_o &= 2\pi r K_r(z = h) = -2\pi a K_z(r = a) = -2\pi r K_r(z = 0) \\ &\Rightarrow K_r(z = h) = \frac{b}{r}K_o;\, K_z(r = a) = \frac{b}{a}K_o;\, K_r(z = 0) = \frac{b}{r}K_o \end{aligned} \tag{2}$$

Note that these surface current densities are what is called for in Ampère's continuity condition, (1.4.16), if the magnetic field given by (1) is to be confined to the annular region.

(c) Faraday's integral law

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_S \mu_o\mathbf{H}\cdot d\mathbf{a} \tag{3}$$

applied to the surface S of Fig. P3.4.2 gives

$$h[E_z(r) - E_z(r=a)] = -\mu_o h \int_r^a H_\phi dr = -\mu_o hb ln(a/r) \frac{dK_o}{dt} \tag{4}$$

Because $E_z(r=a) = 0$, the magnetoquasistatic electric field that goes with (2) in the annular region is therefore

$$E_z = -\mu_o b ln(a/r) \frac{dK_o}{dt} \tag{5}$$

(d) Again, using Ampère's integral law with the contour of Fig. 3.4.2, but this time including the displacement current associated with the time varying electric field of (5), gives

$$2\pi r H_\phi = 2\pi b K_o(t) + \epsilon_o \frac{\partial}{\partial t} \int_b^r E_z 2\pi r dr \tag{6}$$

Note that the first contribution on the right is due to the integral of J associated with the distributed surface current source while the second is due to the displacement current density. Solving (6) for the magnetic field with E_z given by (5) now gives

$$H_\phi = \frac{b}{r} K_o(t) + \frac{\epsilon_o \mu_o b a^2}{r} \left\{ \left(\frac{r}{a}\right)^2 \left[\frac{1}{2} ln\left(\frac{r}{a}\right) - \frac{1}{4}\right] - \left(\frac{b}{a}\right)^2 \left[\frac{1}{2} ln\left(\frac{b}{a}\right) - \frac{1}{4}\right] \right\} \frac{d^2 K_o}{dt^2} \tag{7}$$

The last term is the correction to the magnetoquasistatic approximation. Thus, the MQS approximation is appropriate provided that at $r = a$

$$\frac{H_{\text{correction}}}{(b/a)|K_o|} = \epsilon_o \mu_o a^2 \left\{ \frac{1}{4}\left[\left(\frac{b}{a}\right)^2 - 1\right] - \frac{1}{2}\left(\frac{b}{a}\right)^2 ln\left(\frac{b}{a}\right) \right\} \frac{1}{|K_o|} \left| \frac{d^2 K_o}{dt^2} \right| \tag{8}$$

(e) In the sinusoidal steady state, (8) becomes

$$\frac{H_{\text{correction}}}{M_{\text{MQS}}} = \left(\frac{a}{c}\right)^2 \left| \frac{1}{4}\left[\left(\frac{b}{a}\right)^2 - 1\right] - \frac{1}{2}\left(\frac{b}{a}\right)^2 ln\left(\frac{b}{a}\right) \right| \omega^2 \ll 1 \tag{9}$$

The term in $|\quad|$ is of the order of unity or smaller. Thus, the MQS approximation holds if the electromagnetic delay time a/c is short compared to the reciprocal typical time $1/\omega$.