# SOLUTIONS TO CHAPTER 14

## 14.1 DISTRIBUTED PARAMETER EQUIVALENTS AND MODELS

**14.1.1** The fields are approximated as uniform in each of the dielectric regions. The integral of **E** between the electrodes must equal the applied voltage and $D$ is continuous at the interface. Thus,

$$E_a a + E_b b = -V; \qquad \epsilon_a E_a = \epsilon_b E_b \tag{1}$$

and it follows that

$$E_a = -\frac{V}{[a + b(\epsilon_a/\epsilon_b)]} \tag{2}$$

so that the charge per unit length on the upper electrodes is

$$\lambda_l = -w\epsilon_a E_a = CV; \qquad C \equiv \frac{w\epsilon_a}{[a + b(\epsilon_a/\epsilon_b)]} \tag{3}$$

where $C$ is the desired capacitance per unit length. Because the permeability of the region is uniform, $H = I/w$ between the electrodes. Thus,

$$\lambda = (a + b)\mu_o H = LI; \qquad L \equiv \mu_o(a + b)/w \tag{4}$$

where $L$ is the inductance per unit length. Note that $LC \neq \mu\epsilon$ (which permittivity) unless $\epsilon_a = \epsilon_b$.

**14.1.2** The currents at the node must sum to zero, with that through the inductor related to the voltage by $V = Ldi_{\text{conductor}}/dt$

$$\frac{L}{\Delta z}\frac{\partial}{\partial t}[I(z) - I(z + \Delta z)] = V \tag{1}$$

and $C$ times the rate of change of the voltage drop across the capacitor must be equal to the current through the capacitor.

$$\frac{C}{\Delta z}\frac{\partial}{\partial t}[V(z) - V(z + \Delta z)] = I \tag{2}$$

In the limit where $\Delta z \to 0$, these become the given backward-wave transmission line equations.

## 14.2 TRANSVERSE ELECTROMAGNETIC WAVES

**14.2.1** (a) From Ampère's integral law, (1.4.10),

$$H_\phi = I/2\pi r \tag{1}$$

and the vector potential follows by integration

$$H_\phi = -\frac{1}{\mu_o}\frac{\partial A_z}{\partial r} \Rightarrow A_z(r) - A_z(a) = -\frac{\mu_o I}{2\pi} ln(r/a) \tag{2}$$

and evaluating the integration coefficient by using the boundary condition on $A_z$ on the outer conductor, where $r = a$. The electric field follows from Gauss' integral law, (1.3.13),

$$E_r = \lambda_l/2\pi\epsilon r \tag{3}$$

and the potential follows by integrating.

$$E_r = -\frac{\partial\Phi}{\partial r} \Rightarrow \Phi(r) - \Phi(a) = \frac{\lambda_l}{2\pi\epsilon} ln\left(\frac{a}{r}\right) \tag{4}$$

Using the boundary condition at $r = a$ then gives the potential.

(b) The inductance per unit length follows from evaluation of (2) at the inner boundary.

$$L \equiv \frac{\Lambda}{I} = \frac{A_z(b) - A_z(a)}{I} = \frac{\mu_o}{2\pi} ln(a/b) \tag{5}$$

Similarly, the capacitance per unit length follows from evaluating (5) at the inner boundary.

$$C \equiv \frac{\lambda_l}{V} = \frac{2\pi\epsilon}{ln(a/b)} \tag{6}$$

**14.2.2** The capacitance per unit length is as given in the solution to Prob. 4.7.5. The inductance per unit length follows by using (8.6.14), $L = 1/Cc^2$.

## 14.3 TRANSIENTS ON INFINITE TRANSMISSION LINES

**14.3.1** (a) From the values of $L$ and $C$ given in Prob. 14.2.1, (14.3.12) gives

$$Z_o = \sqrt{\mu/\epsilon} ln(a/b)/2\pi$$

(b) From $\mu = \mu_o = 4\pi \times 10^{-7}$ and $\epsilon = 2.5\epsilon_o = (2.5)(8.8.5 \times 10^{-12}), Z_o = (37.9)ln(a/b)$. Because the only effect of geometry is through the ratio $a/b$ and that is logarithmic, the range of characteristic impedances encoutered in practice for coaxial cables is relatively small, typically between 50 and 100 ohms. For example, for the four ratios of $a/b, Z_o = 26, 87, 175$ and 262 Ohms, respectively. To make $Z_o = 1000$ Ohms would require that $a/b = 2.9 \times 10^{11}$!

**14.3.2** The characteristic impedance is given by (14.3.13). Presuming that we will find that $l/R \gg 1$, the expression is approximated by

$$Z_o = \sqrt{\mu/\epsilon} ln(2l/R)/\pi$$

and solved for $l/R$.

$$l/R = \frac{1}{2}\exp[\pi Z_o/\sqrt{\mu/\epsilon}] = \exp[\pi(300)/377]$$

Evaluation then gives $l/R = 6.1$.

**14.3.3** The solution is analogous to that of Example 14.3.2 and shown in the figure.

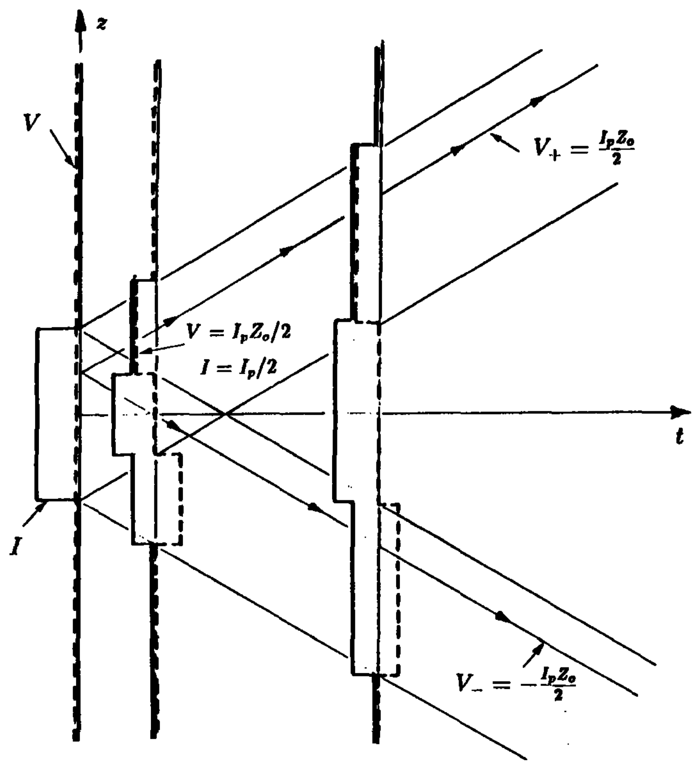


**Figure S14.3.3**

**14.3.4** From (14.3.18) and (14.3.19), it follows that

$$V_\pm = V_o \exp(-z^2/2a^2)/2$$

Then, from (9) and (10)

$$V = \frac{1}{2}V_o\{\exp[-(z-ct)^2/2a^2] + \exp[-(z+ct)^2/2a^2]\}$$

$$I = \frac{1}{2}\frac{V_o}{Z_o}\{\exp[-(z-ct)^2/2a^2] - \exp[-(z+ct)^2/2a^2]\}$$

**14.3.5** In general, the voltage and current can be represented by (14.3.9) and (14.3.10). From these it follows that

$$VI = (V_+ + V_-)\frac{1}{Z_o}(V_+ - V_-) = \frac{1}{Z_o}(V_+^2 - V_-^2)$$

**14.3.6** By taking the $\partial()/\partial t$ and $\partial()/\partial z$ of the second equation in Prob. 14.1.2 and substituting it into the first, we obtain the partial differential equation that plays the role played by the wave equation for the conventional transmission line

$$LC\frac{\partial^4 V}{\partial t^2 \partial x^2} = V \tag{1}$$

Taking the required derivatives on the left amounts to combining (14.3.6). Thus, substitution of (14.3.3) into (1), gives

$$c^2 V_{\pm}'''' = V$$

By contrast with the wave-equation, this expression is not identically satisfied. Waves do not propagate on this line without dispersion.

## 14.4 TRANSIENTS ON BOUNDED TRANSMISSION LINES

**14.4.1** When $t = 0$, the initial conditions on the line are

$$V = V_o; \qquad I = 0 \quad \text{for} \quad 0 < z < l$$

From (14.4.4) and (14.4.5), it follows that for those characteristics originating on the $t = 0$ axis of the figure

$$V_+ = V_o/2; \qquad V_- = V_o/2$$

For those lines originating at $z = l$, it follows from (14.4.8) with $R_L = \infty (\Gamma_L = 1)$ that

$$V_- = V_+$$

Similarly, for those lines originating at $z = 0$, it follows from (14.4.10) with $\Gamma_g = 0$ and $V_g = 0$ that

$$V_+ = 0$$

Combining these invarients in accordance with (14.1.1) and (14.1.2) at each location gives the $(z, t)$ dependence of $V$ and $I$ shown in the figure.

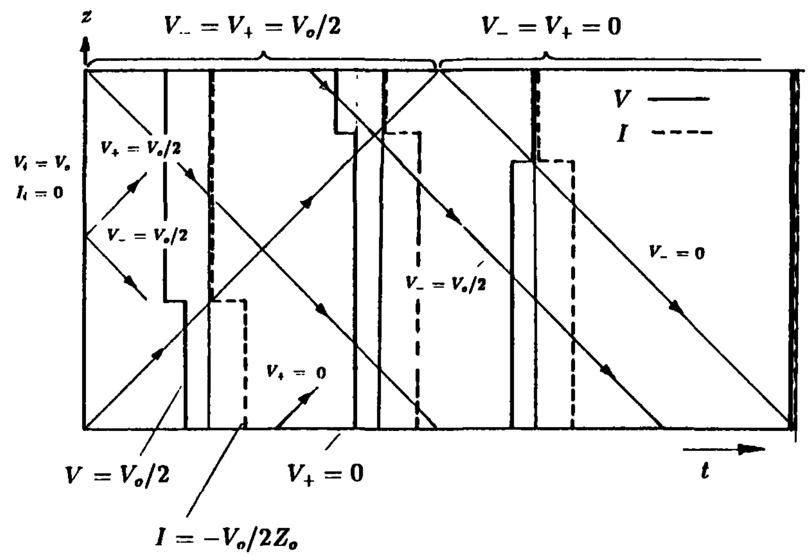


**Figure S14.4.1**

**14.4.2** When $t = 0$, the initial conditions on the line are

$$V = 0; \qquad I = V_o/Z_o \quad \text{for} \quad 0 < z < l$$

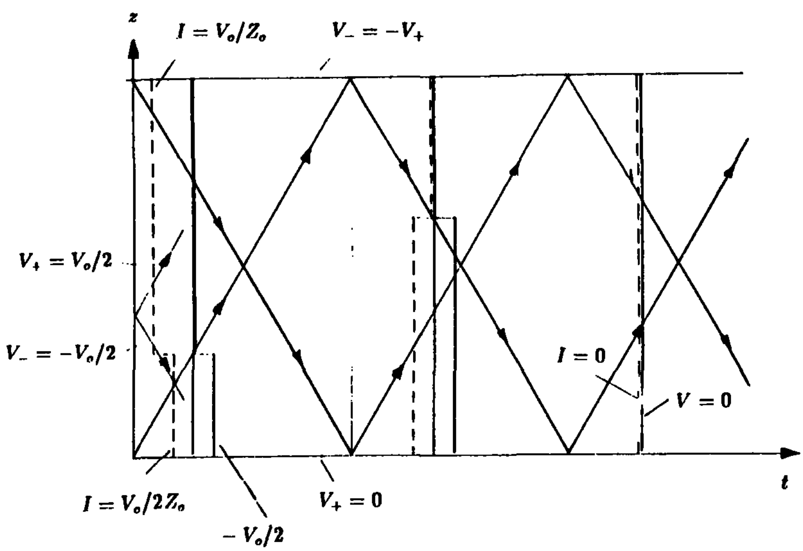


**Figure S14.4.2**

From (14.4.4) and (14.4.5), it follows that for those characteristics originating on the $t = 0$ axis of the figure

$$V_+ = V_o/2; \qquad V_- = -V_o/2$$

For those lines originating at $z = l$, it follows from (14.4.8) with $R_L = 0$ ($\Gamma_L = -1$) that

$$V_- = -V_+$$

Similarly, for those lines originating at $z = 0$, it follows from (14.4.10) that

$$V_+ = 0$$

Combining these invarients in accordance with (14.4.1) and (14.4.2) at each location gives the $(z, t)$ dependence of $V$ and $I$ shown in the figure.

**14.4.3** If the voltage and current on the line are initially zero, then it follows from (14.4.5) that $V_- = 0$ on those characteristic lines $x + ct =$ constant that originate on the $t = 0$ axis. Because $R_L = Z_o$, it follows from (14.4.8) that $V_- = 0$ for all of the other lines $x + ct =$ constant, which originate at $z = l$. Thus, at $z = 0$, (14.4.1) and (14.4.2) become

$$V = V_+; \qquad I = V_+/Z_o$$

and the ratio of these is the terminal relation $V/I = Z_o$, the relation for a resistance equal in value to the characteristic impedance. Implicit to this equivalence is the condition that the initial voltage and current on the line be zero.

**14.4.4**

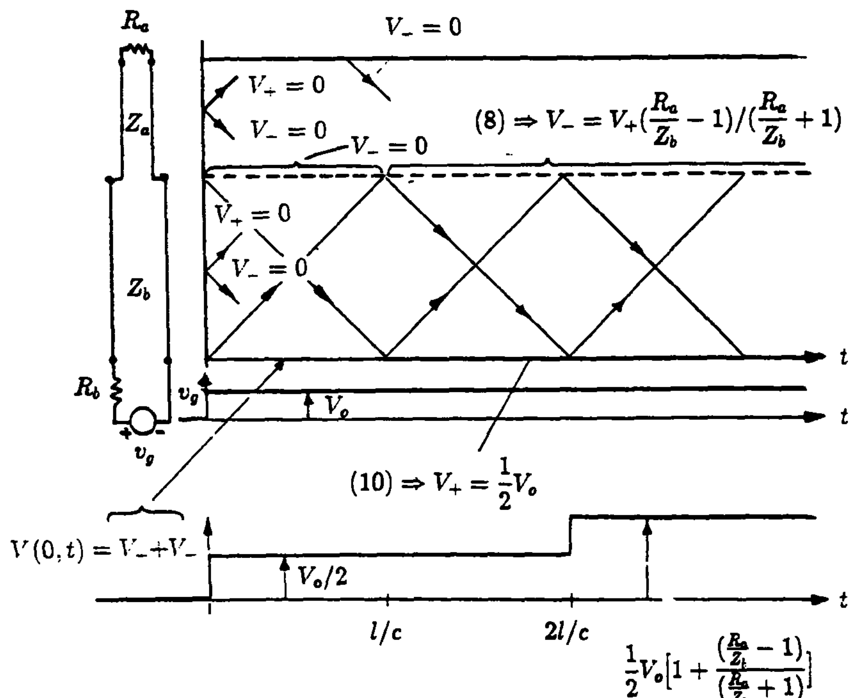


**Figure S14.4.4**

The solution is constructed in the $x-t$ plane as shown by the figure. Because the upper transmission line is both terminated in its characteristic impedance and free of initial conditions, it is equivalent to a resistance $R_a$ connected to the terminals of the lower line (see Prob. 14.4.3). The values of $V_+$ and $V_-$ follow from (14.4.4) and (14.4.5) for the characteristic lines originating when $t = 0$ and from (14.4.8) and (14.4.10) for those respectively originating at $z = l$ and $z = 0$.

**14.4.5** When $t < 0$, a steady current flows around the loop and the initial voltage and current distribution are uniform over the length of the two line-segments.

$$V_i = \frac{R_a V_o}{R_a + R_b}; \qquad I_i = \frac{V_o}{R_a + R_b}$$

In the upper segment, shown in the figure, it follows from (14.4.5) that $V_- = 0$. Thus, for these particular initial conditions, the upper segment is equivalent to a termination on the lower segment equal to $Z_a = R_a$. In the lower segment, $V_+$ and $V_-$ originating on the $z$ axis follow from the initial conditions and (14.4.4) and (14.4.5) as being the values given on the $z-t$ diagram. The conditions relating the incident to the reflected waves, given respectively by (14.4.8) and (14.4.10), are also summarized in the diagram. Use of (14.4.1) to find $V(0,t)$ then gives the function of time shown at the bottom of the figure.

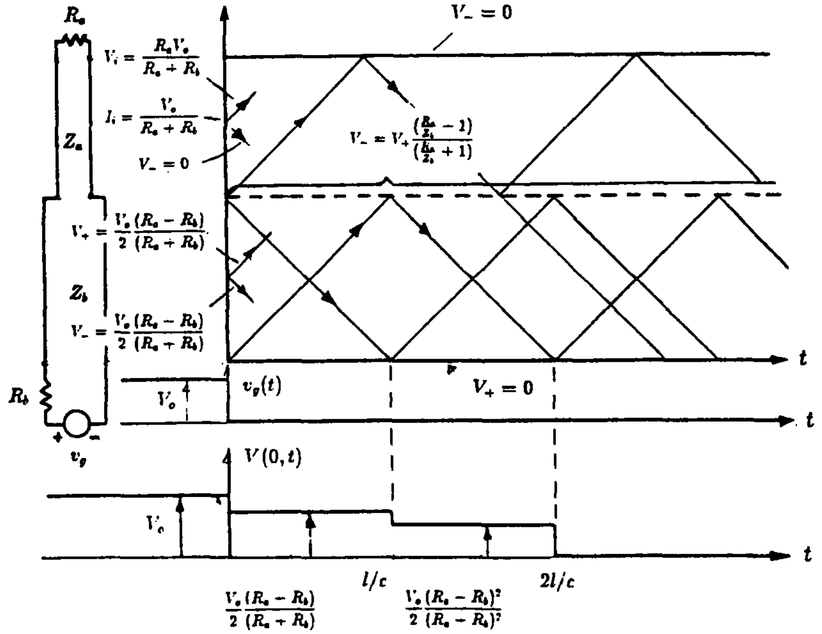


**Figure S14.4.5**

**14.4.6** From (14.4.4) and (14.4.5), it follows from the initial conditions that $V_+$ and $V_-$ are zero on lines originating on the $t = 0$ axis. The value of $V_+$ on lines coming from the $z = 0$ axis is determined by requiring that the currents at the input terminal sum to zero.

$$i_g = \frac{1}{R_g}(V_+ + V_-) + \frac{1}{Z_o}(V_+ - V_-)$$

or

$$V_+ = \frac{i_g}{(1/R_g + 1/Z_o)} + V_- \frac{(1/Z_o - 1/R_g)}{(1/Z_o + 1/R_g)}$$

It follows that for $0 < t < T, V_+ = I_o R_g/2$ while for $T < t, V_+ = 0$. At $z = l$, (14.4.8) shows that $V_+ = -V_-$. Thus, the solution is as summarised in the figure.

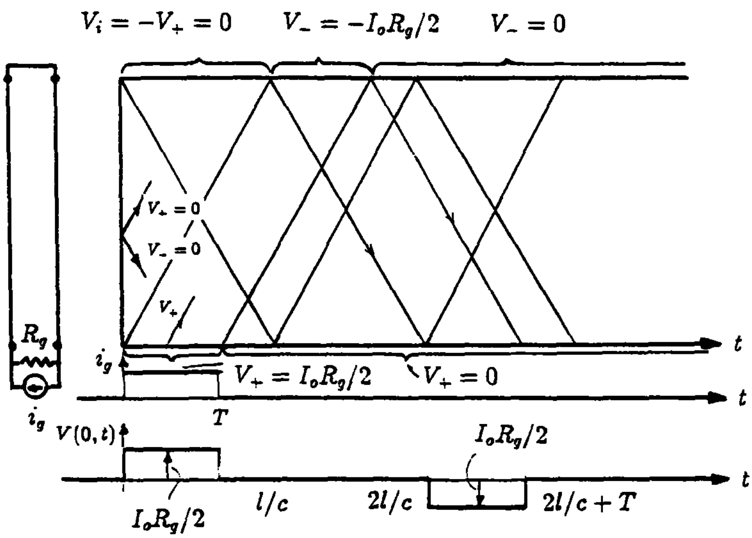


**Figure S14.4.6**

14.4.7 (a) By replacing $V_+/Z_o \rightarrow I_+, V_-/Z_o \rightarrow -I_-$, the general solutions given by (14.4.1) and (14.4.2) are written in terms of currents rather than voltages.

$$I = I_+ + I_-; \qquad V = \frac{1}{Y_o}(I_+ - I_-) \tag{1}$$

where $Y_o \equiv 1/Z_o$. When $t = 0$, the initial conditions are zero, so on characteristic lines originating on the $t = 0$ axis, $I_+$ and $I_-$ are zero. At $z = l$, it follows from (1b) that $I_+ = I_-$. At $z = 0$, summation of currents at the terminal gives

$$i_g = (G_g/Y_o)(I_+ - I_-) + (I_+ + I_-) \tag{2}$$

which, solved for the reflected wave in terms of the incident wave gives

$$I_+ = I_g + I_- \Gamma_g \tag{3}$$

where

$$I_g \equiv \frac{I_o}{1+(G_g/Y_o)}; \qquad \Gamma_g \equiv [(G_g/Y_o)-1]/[(G_g/Y_o)+1] \tag{4}$$

From these relations, the wave components $I_+$ and $I_-$ are constructed as summarized in the figure. The voltage at the terminals of the line is

$$V(0,t) = \frac{I_o}{G_g[1+(Y_o/G_g)]}\Gamma_g^{N-1}; \quad 2(N-1)\frac{l}{c} < t < 2N\frac{l}{c} \tag{5}$$

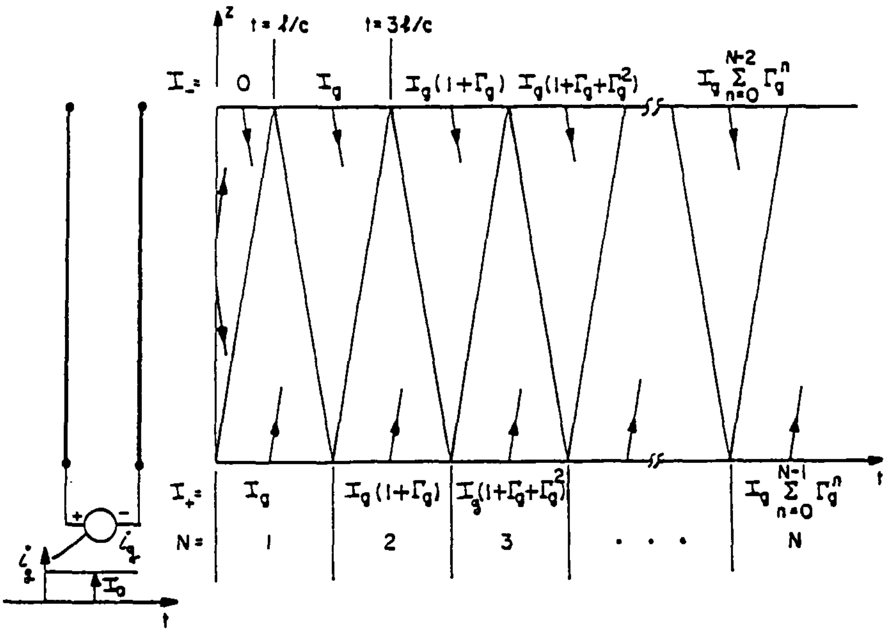


**Figure S14.4.7**

It follows that during this same interval, the terminal current is

$$I(0,t) = I_o\left(1 - \frac{\Gamma_g^{N-1}}{(1+Y_o/G_g)}\right) \tag{6}$$

(b) In terms of the terminal current $I$, the circuit equation for the line in the limit where it behaves as an inductor is

$$i_g = lLG_g\frac{dI}{dt} + I \tag{7}$$

Solution of this expression with $i_g = I_o$ and $I(0) = 0$ is

$$I(0,t) = I_o(1 - e^{-t/\tau}); \quad \tau \equiv lLG_g \tag{8}$$

(c) In the limit where $G_g/Y_o$ is very large

$$\Gamma_g \to 1 - 2(Y_o/R_g) \tag{9}$$

Thus,

$$I(0,t) \to I_o\Big[1 - \big(1 - \frac{2Y_o}{G_g}\big)^{N-1}\Big]; \quad 2(N-1)\frac{l}{c} < t < \frac{2Nl}{c} \tag{10}$$

Following the same arguments as given by (14.4.28)–(14.4.31), gives

$$I(0,t) \to I_o[1 - e^{-(N-1)(2Y_o/G_g)}]; \quad 2(N-1)\frac{l}{c} < t < 2N\frac{l}{c} \tag{11}$$

which in the limit here, (14.4.31) holds the same as (8) where $(Y_o/G_g)(c/l) = \sqrt{C/L}/l\sqrt{LC}G_g = 1/lLG_g$. Thus, the current reponse (which has the same stair-step dependence on time as for the analogous example represented by Fig. 14.4.8) becomes the exponential response of the circuit in the limit where the inductor takes a long time to "charge" compared to the transit-time of an electromagnetic wave.

14.4.8

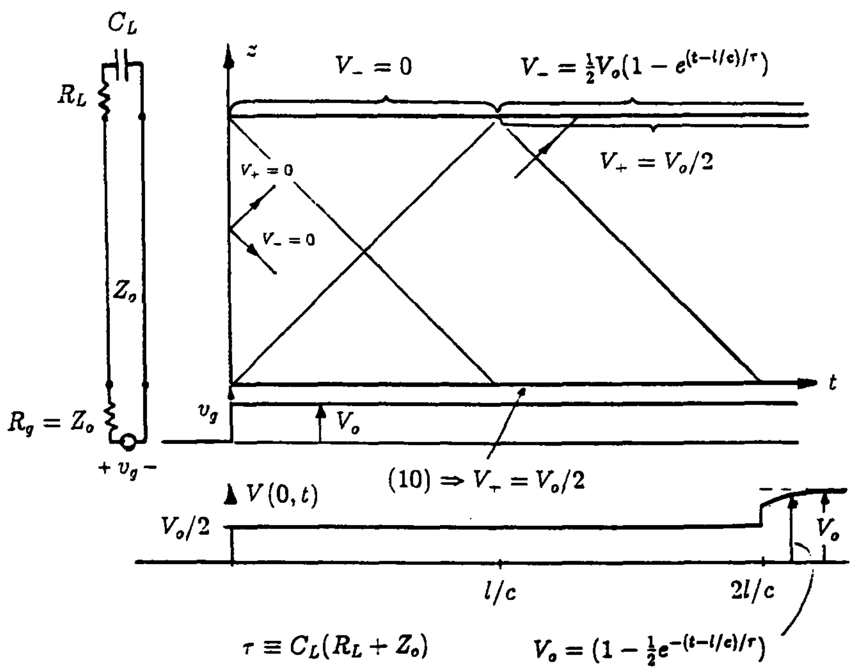


**Figure S14.4.8**

The initial conditions on the voltage and current are zero and it follows from (14.4.4) and (14.4.5) that $V_+$ and $V_-$ on characteristics originating on the $t = 0$

axis are zero. It follows from (10) that on the lines originating on the $z = 0$ axis, $V_+ = V_o/2$. Then, for $0 < t < l/c$, the incident $V_+$ at $z = l$ is zero and hence from the differential equation representing the load resistor and capacitor, it follows that $V_- = 0$ during this time as well. For $l/c < t, V_+ = V_o/2$ at $z = l$. In view of the steady state established while $t < 0$, the initial capacitor voltage is zero. Thus, the initial value of $V_-(l, 0)$ is zero and the reflected wave is predicted by

$$C_L(R_L + Z_o)\frac{dV_-}{dt} + V_- = \frac{V_o}{2}u_{-1}(t - l/c)$$

The appropriate solution is

$$V_-(l, t) = \frac{1}{2}V_o(1 - e^{-(t-l/c)/\tau}); \quad \tau \equiv C_L(R_L + Z_o)$$

This establishes the wave incident at $z = 0$. The solution is summarized in the figure.

## 14.5 TRANSMISSION LINES IN THE SINUSOIDAL STEADY STATE

**14.5.1** From (14.5.20), for the load capacitor where $Z_L = 1/j\omega C_L$,

$$\frac{Y(\beta l = -\pi/2)}{Y_o} = \frac{Y_o}{Y_L} = \frac{Y_o}{j\omega C_L}$$

Thus, the impedance is inductive.

For the load inductor where $Z_L = j\omega L_L$, (14.5.20) gives

$$\frac{Z(\beta l = -\pi/2)}{Z_o} = \frac{Z_o}{j\omega L_L}$$

and the impedance is capacitive.

**14.5.2** For the open circuit, $Z_L = \infty$ and from (14.5.13), $\Gamma_L = 1$. The admittance at any other location is given by (14.5.10).

$$\frac{Y(-l)}{Y_o} = \frac{1 - \Gamma_L e^{-2j\beta l}}{1 + \Gamma_L e^{-2j\beta l}} = \frac{1 - e^{-2j\beta l}}{1 + e^{-2j\beta l}}$$

where characteristic admittance $Y_o = 1/Z_o$. This expression reduces to

$$\frac{Y(-l)}{Y_o} = j \tan \beta l$$

which is the same as the impedance for the shorted line, (14.5.17). Thus, with the vertical axis the admittance normalized to the characteristic admittance, the frequency or length dependence is as shown by Fig. 14.5.2.

**14.5.3** The matched line requires that $\hat{V}_- = 0$. Thus, from (14.5.5) and (14.5.6),

$$\hat{V} = \hat{V}_+ \exp(-j\beta z); \qquad \hat{I} = \frac{\hat{V}_+}{Z_o}\exp(-j\beta z)$$

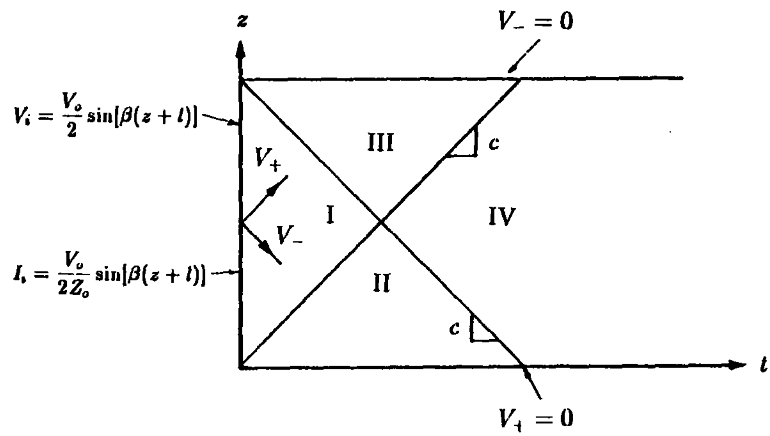


**Figure S14.5.3**

At $z = -l$, the circuitis described by

$$\hat{V}_g = \hat{I}(-l)R_g + \hat{V}(-l)$$

where, in complex notation, $V_g = \text{Re}\,\hat{V}_g \exp(j\omega t)$, $\hat{V}_g \equiv -jV_o$. Thus, for $R_g = Z_o$,

$$\hat{V}_g = \frac{R_g}{Z_o}\hat{V}_+ e^{j\beta l} + \hat{V}_+ e^{j\beta l} \Rightarrow \hat{V}_+ = \frac{1}{2}\hat{V}_g e^{-j\beta l}$$

and the given sinusoidal steady state solutions follow.

$$V = \text{Re}\,\frac{-jV_o}{2}e^{-j\beta(z+l)}e^{j\omega t}; \quad I = \text{Re}\,\frac{-jV_o}{2Z_o}e^{-j\beta(z+l)}e^{j\omega t}$$

**14.5.4** Initially, both the current and voltage are zero. With the solution written as the sum of the sinusoidal steady state solution found in Prob. 14.5.3 and a transient solution,

$$V = V_s(z,t) + V_t(z,t); \qquad I = I_s(z,t) + I_t(z,t)$$

the initial conditions on the transient part are therefore,

$$V_t(z,0) = -V_s(z,0) = \frac{V_o}{2}\sin[\beta(z+l)]$$

$$I_t(z,0) = -I_s(z,0) = \frac{V_o}{2Z_o}\sin[\beta(z+l)]$$

The boundary conditions for $0 < t$ and with the given driving source are satisfied by $V_s$. Thus, $V_t$ must satisfy the boundary conditions that result if $V_g = 0$. In terms of a transient solution written as 14.3.9 and 14.3.10, these are that $V_- = 0$ at $z = 0$ and [from (14.5.10) with $V_g = 0$ and $R_g = Z_o$] that $V_+ = 0$ at $z = -l$. Thus, the initial and boundary conditions for the transient part of the solution are as summarized in the figure. With the regions in the $x - t$ plane denoted as shown in the figure, the voltage and current are therefore,

$$V = V_s + V_t; \qquad I = I_s + I_t$$

where $V_s$ and $I_s$ are as given in Prob. 14.5.3 and

$$V_t = V_+ + V_-; \qquad I_t = \frac{1}{Z_o}(V_+ - V_-)$$

with (from 14.3.18-19)

$$V_+ = \frac{V_o}{2}\sin[\beta(z+l)]; \qquad V_- = 0$$

in regions I and III, and

$$V_+ = 0; \qquad V_- = 0$$

in regions II and IV.

## 14.6 REFLECTION COEFFICIENT REPRESENTATION OF TRANSMISSION LINES

**14.6.1** The Smith chart solution is like the case of the Quarter-Wave Section exemplified using Fig. 14.6.3. The load is at $r = 2, x = 2$ on the chart. The impedance a quarater-wave toward the generator amounts to a constant radius clockwise rotation of 180° to the point where $r = 0.25$ and $x = -j0.25$. Evaluation of (14.6.20) checks this result, because it shows that

$$r + jx\big|_{z=-l} = \frac{1}{r_L + jx_L} = \frac{1}{2+j2} = \frac{2-j2}{8}$$

**14.6.2** From (14.6.3), $\Gamma = 0.538 + j0.308$ and $|\Gamma| = 0.620$. It follows from (14.6.10) that the VSWR is 4.26. These values also follow from drawing a circle through $r + jx = 2 + j2$, using the radius of the circle to obtain $|\Gamma|$ and the construction of Fig. 14.6.4a to evaluate (14.6.10).

**14.6.3** The angular distance on the Smith charge from the point $y = 2 + j0$ to the circle where $y$ has a real part of 1 is $l = 0.0975\lambda$. To cancel the reactance, where $y = 1 + j0.7$ at this point, the distance from the shorted end of the stub to the point where it is attached to the line must be $l_s = 0.347\lambda$.

**14.6.4** Adjustment of the length of the first stub makes it possible to be anywhere on the circle $g = 2$ of the admittance chart at the terminals of the parallel stub and load. If this admittance can be transferred onto the circle $g = 1$ by moving a distance $l$ toward the generator (clockwise), the second stub can be used to match the line by compensating for the reactive part of the impedance. Thus, determination of the stub lengths amounts to finding a pair of points on these circles that are at the same radius and separated by the angle $0.042\lambda$. This then gives both the combined stub (1) and load impedance (for the case given, $y = 2 + j1.3$) and combined stub (2) and line impedance at $z = -l$ (for the case given, $y = 1 + j1.16$). To create the needed susceptance at the load, $l_1 = 0.04\lambda$. To cancel the resulting susceptance at the second stub, $l_2 = 0.38\lambda$.

**14.6.5** The impedance at the left end of the quarter wave section is 0.5. Thus, normalized to the impedance of the line to the left, the impedance there is $Z/Z_o^a = 0.25$. It follows from the Smith chart and (14.6.10) that the VSWR = 4.0.

## 14.7 DISTRIBUTED PARAMETER EQUIVALENTS AND MODELS WITH DISSIPATION

**14.7.1** The currents must sum to zero at the node. With those through the conductance and capacitance on the right,

$$I(z) - I(z + \Delta z) = G\Delta z V + C\Delta z \frac{\partial V}{\partial t}$$

The voltage drop around a loop comprised of the terminals and the series resistance and inductance must sum of zero. With the voltage drops across the resistor and inductor on the right,

$$V(z) - V(z + \Delta z) = R\Delta z I + L\Delta z \frac{\partial I}{\partial t}$$

In the limit where $\Delta z \to 0$, these expressions become the transmission line equations, (14.7.1) and (14.7.2).

**14.7.2** (a) If the voltage is given by (14.7.12), as a special case of (14.7.9), then it follows that $I(z,t)$ is the special case of (14.7.10)

$$I = \text{Re}\, \frac{\hat{V}_g}{Z_o} \frac{\left(e^{-j\beta z} - e^{j\beta z}\right)}{\left(e^{j\beta l} + e^{-j\beta l}\right)} e^{j\omega t}$$

(b) The desired impedance is the ratio of the voltage, (14.7.12), to this current, evaluated at $z = -l$.

$$Z = Z_o \frac{(e^{j\beta l} + e^{-j\beta l})}{(e^{j\beta l} - e^{-j\beta l})}$$

(c) In the long-wave limit, $|\beta l| \ll 1$, $\exp(j\beta l) \rightarrow 1 + j\beta l$ and this expression becomes

$$Z = \frac{Z_o}{j\beta l} = \frac{(R + j\omega L)}{-\beta^2 l} = \frac{1}{[G + j\omega C]l}$$

where (14.7.8) and (14.7.11) have been used to write the latter equality. (Note that (14.7.8) is best left in the form suggested by (14.7.7) to obtain this result.) The circuit having this impedance is a conductance $lG$ shunted by a capacitance $lC$.

**14.7.3** The short requires that $V(0,t) = 0$ gives $V_+ = V_-$. With the magnitude adjusted to match the condition that $V(-l,t) = V_g(t)$, (14.7.9) and (14.7.10) become

$$\hat{V} = \hat{V}_g \frac{(e^{-j\beta z} - e^{j\beta z})}{(e^{j\beta l} - e^{-j\beta l})}; \quad I = \text{Re}\, \frac{\hat{V}_g}{Z_o} \frac{(e^{-j\beta z} + e^{j\beta z})}{(e^{j\beta l} - e^{-j\beta l})}$$

Thus, the impedance at $z = -l$ is

$$Z = Z_o(e^{j\beta l} - e^{-j\beta l})/(e^{j\beta l} + e^{-j\beta l})$$

In the limit where $|\beta l| \ll 1$, it follows from this expression and (14.7.8) and (14.7.11) that because $\exp j\beta l \rightarrow 1 + j\beta l$

$$Z \rightarrow Z_o j\beta l = l(R + j\omega L)$$

which is the impedance of a resistance $lR$ in series with an inductor $lL$.

**14.7.4** (a) The theorem is obtained by adding the negative of $V$ times (1) to the negative of $I$ times (2).

(b) The identity follows from

$$\begin{aligned}\text{Re}\, \hat{A}e^{j\omega t}\text{Re}\, \hat{B}e^{j\omega t} &= \frac{1}{2}[\hat{A}e^{j\omega t} + \hat{A}^* e^{-j\omega t}]\frac{1}{2}[\hat{B}e^{j\omega t} + \hat{B}^* e^{-j\omega t}] \\ &= \frac{1}{4}[\hat{A}\hat{B}e^{2j\omega t} + \hat{A}^*\hat{B}^* e^{-2j\omega t}] + \frac{1}{4}[\hat{A}\hat{B}^* + \hat{A}^*\hat{B}] \\ &= \text{Re}\, \frac{1}{2}\hat{A}\hat{B}e^{2j\omega t} + \text{Re}\, \frac{1}{2}\hat{A}\hat{B}^*\end{aligned}$$

(c) Each of the quadratic terms in the power theorem take the form of (1), a time independent part and a part that varies sinusoidally at twice the driving frequency. The periodic part time-averages to zero in the power flux term on the left and in the dissipation terms (the last two terms) on the right. The only contribution to the energy storage term is due to the second harmonic, and that time-averages to zero. Thus, on the time-average there is no contribution from the energy storage terms.

The integral theorem, (d) follows from the integration of (c) over the length of the system. Integration of the derivative on the left results in the integrand evaluated at the end points. Because the current is zero where $z = 0$, the only contribution is the time-average input power on the left in (d).

(d) The left hand side is evaluated using (14.7.12) and (14.7.6). First, using (14.7.11), (14.7.6) becomes

$$\hat{I} = -Y_o \hat{V}_g \tan \beta z \tag{2}$$

Thus,

$$\frac{1}{2}\text{Re}\,\hat{V}\hat{I}^*\Big|_{z=-l} = \frac{1}{2}\text{Re}\,\hat{V}_g^*\hat{I}\Big|_{z=-l} = \frac{1}{2}\text{Re}\,|\hat{V}_g|^2 Y_o \tan \beta l \tag{3}$$

That the right hand side must give the same thing follows from using (14.7.3) and (14.7.4) to write

$$G\hat{V}^* = j\omega C\hat{V}^* - \frac{d\hat{I}^*}{dz} \tag{4}$$

$$R\hat{I} = -\frac{d\hat{V}}{dz} - j\omega L\hat{I} \tag{5}$$

Thus,

$$\begin{aligned}
\int_{-l}^{0} \frac{1}{2}\text{Re}\,[\hat{I}^* RI + \hat{V}\hat{V}^* G]dz &= \int_{-l}^{0} \frac{1}{2}\text{Re}\,[-\hat{I}\frac{d\hat{V}^*}{dz} - j\omega L\hat{I}\hat{I}^* \\
&\quad + j\omega C\hat{V}\hat{V}^* - \hat{V}\frac{d\hat{I}^*}{dz}]dz \\
&= -\int_{-l}^{0} \frac{1}{2}\text{Re}\,[\hat{I}\frac{d\hat{V}^*}{dz} + \hat{V}\frac{d\hat{I}^*}{dz}]dz \\
&= -\frac{1}{2}\text{Re}\int_{-l}^{0} \frac{d(\hat{I}\hat{V}^*)}{dz}dz \\
&= \frac{1}{2}\text{Re}\,\hat{I}\hat{V}^*\Big|_{z=-l}
\end{aligned} \tag{6}$$

which is the same as (3).

## 14.8 UNIFORM AND TEM WAVES IN OHMIC CONDUCTORS

**14.8.1** In Ampère's law, represented by (12.1.4), $\mathbf{J}_u = \sigma\mathbf{E}$. Hence, (12.1.6) becomes

$$\nabla(\nabla\cdot\mathbf{A} + \mu\sigma\Phi + \mu\epsilon\frac{\partial\Phi}{\partial t}) - \nabla^2\mathbf{A} = -\mu\sigma\frac{\partial\mathbf{A}}{\partial t} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} \tag{1}$$

Hence, the gauge condition, (14.8.3), becomes

$$\nabla\cdot\mathbf{A} = \frac{\partial A_z}{\partial z} = -\mu\sigma\Phi - \mu\epsilon\frac{\partial\Phi}{\partial t} \tag{2}$$

Evaluation of this expression on the conductor surface with (14.8.9) and (14.8.11) gives

$$L\frac{\partial I}{\partial z} = -\mu\sigma V - LC\frac{\partial V}{\partial t} \tag{3}$$

From (8.6.14) and (7.6.4)

$$\frac{\mu\sigma}{L} = \frac{\mu\sigma C}{\mu\epsilon} = \frac{G}{C}C = G \tag{4}$$

Thus,

$$\frac{\partial I}{\partial z} = -GV - C\frac{\partial V}{\partial t} \tag{5}$$

This and (14.8.12) are the desired transmission line equations including the losses represented by the shunt conductance $G$. Note that, provided the conductors are "perfect", the TEM wave represented by these equations is exact and not quasi-one-dimensional.

**14.8.2** The transverse dependence of the electric and magnetic fields are respectively the same as for the two-dimensional EQS capacitor-resistor and MQS inductor. The axial dependence of the fields is as given by (14.8.10) and (14.8.11). Thus, with (Prob. 14.2.1)

$$C = 2\pi\epsilon/ln(a/b); \quad L = \frac{\mu_o}{2\pi}ln(a/b); \quad G = \frac{\sigma}{\epsilon}C = 2\pi\sigma/ln(a/b)$$

and hence $\beta$ and $Z_o$ given by (14.7.8) and (14.7.11) with $R = 0$, the desired fields are

$$\mathbf{E} = \text{Re}\,\frac{\hat{V}_g(e^{-j\beta z} + e^{j\beta z})}{r ln(a/b)(e^{j\beta l} + e^{-j\beta l})}e^{j\omega t}\mathbf{i}_r$$

$$\mathbf{H} = \text{Re}\,\frac{\hat{V}_g(e^{-j\beta z} - e^{j\beta z})}{2\pi r Z_o(e^{j\beta l} + e^{-j\beta l})}e^{j\omega t}\mathbf{i}_\phi$$

**14.8.3** The transverse dependence of the potential follows from (4.6.18)–(4.6.19), (4.6.25) and (4.6.27). Thus, with the axial dependence given by (14.8.10),

$$\mathbf{E} = -\frac{\partial \Phi}{\partial x}\mathbf{i_x} - \frac{\partial \Phi}{\partial y}\mathbf{i_y}$$

where

$$\Phi = -\mathrm{Re}\,\frac{\hat{V}_g}{2}\,\frac{ln\left[\frac{\sqrt{(\sqrt{l^2-R^2}-x)^2+y^2}}{\sqrt{(\sqrt{l^2-R^2}+x)^2+y^2}}\right]}{ln\left[\frac{l}{R}+\sqrt{(l/R)^2-1}\right]}\,\frac{(e^{-j\beta z}+e^{j\beta z})}{(e^{j\beta l}+e^{j\beta l})}e^{j\omega t}$$

Using (14.2.2) $A_z$ follows from this potential.

$$\mathbf{H} = \frac{1}{\mu_o}\left(\frac{\partial A_z}{\partial y}\mathbf{i_x} - \frac{\partial A_z}{\partial x}\mathbf{i_y}\right)$$

where

$$A_z = -\mathrm{Re}\,\frac{\mu_o \hat{V}_g}{2\pi Z_o}\,ln\frac{\sqrt{(\sqrt{l^2-R^2}-x)^2+y^2}}{\sqrt{(\sqrt{l^2-R^2}+x)^2+y^2}}\,\frac{(e^{-j\beta z}-e^{j\beta z})}{(e^{j\beta l}+e^{-j\beta l})}e^{j\omega t}$$

In these expressions, $\beta$ and $Z_o$ are evaluated from (14.7.8) and (14.7.11) using the values of $C$ and $L$ given by (4.6.27) and (4.6.12) with $R = 0$ and $G = (\sigma/\epsilon)C$.

**14.8.4** (a) The integral of $\mathbf{E}$ around the given contour is equal to the negative rate of change of the magnetic flux linked. Thus,

$$aE^a(z+\Delta z) - aE^a(z) + bE^b(z+\Delta z) - bE^b(z) = -(a+b)\Delta z\frac{\partial}{\partial t}(\mu_o H_y) \quad (1)$$

and in the limit $\Delta z \to 0$,

$$a\frac{\partial E^a}{\partial z} + b\frac{\partial E^b}{\partial z} = -\mu_o(a+b)\frac{\partial H_y}{\partial t} \quad (2)$$

Because $\epsilon_a E^a = \epsilon_b E^b$, this expression becomes

$$\left(a + \frac{\epsilon_a}{\epsilon_b}b\right)\frac{\partial E^a}{\partial z} = -\mu_o(a+b)\frac{\partial H_y}{\partial t} \quad (3)$$

If $E^a$ and $H_y$ were to be respectively written in terms of $V$ and $I$, this would be the transmission line equation representing the law of induction (see Prob. 14.1.1).

(b) A similar derivation using the contour closing at the interface gives

$$aE^a(z+\Delta z) - aE^a(z) + \Delta z E_z = -a\mu_o\Delta z\frac{\partial H_y}{\partial t} \quad (4)$$

and in the limit $\Delta z \to 0$,

$$E_z = -a\mu_o \frac{\partial H_y}{\partial t} - a\frac{\partial E^a}{\partial z} \tag{5}$$

With the use of (3), this expression becomes

$$E_z = -\left[a\mu_o\left(\frac{\epsilon_a}{\epsilon_b} - 1\right)b/\left(a + \frac{\epsilon_a}{\epsilon_b}b\right)\right]\frac{\partial H_y}{\partial t} \tag{6}$$

Finally, for a wave having a $z$ dependence $\exp(-j\beta z)$, the desired ratio follows from (6) and (3).

$$\frac{|E_z|}{|E_a|} = \frac{b(\beta a)}{a+b}\left|1 - \frac{\epsilon_a}{\epsilon_b}\right| \tag{7}$$

Thus, the approximation is good provided the wavelength is large compared to $a$ and $b$ and is exact in the limit where the dielectric is uniform.

## 14.9 QUASI-ONE-DIMENSIONAL MODELS

**14.9.1** From (14.9.11)

$$R = \frac{2}{\pi R^2 \sigma}$$

while, from (4.7.2) and (8.6.12) respectively

$$C = \frac{2\pi\epsilon}{ln\left[(l/a) + \sqrt{(l/a) - 1}\right]}; \qquad L = \frac{\mu}{\pi} ln\left[(l/a) + \sqrt{(l/a)^2 - 1}\right]$$

To make the skin depth small compared to the wire radius

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \gg R \Rightarrow \omega \ll 2/a^2\mu\sigma$$

For the frequency to be high enough that the inductive reactance dominates

$$\frac{\omega L}{R} = \frac{\omega\mu\sigma a^2}{2} ln\left[(l/a) + \sqrt{(l/a)^2 - 1}\right]$$

Thus, the frequency range over which the inductive reactance dominates but the constant resistance model is still appropriate is

$$\frac{2}{\mu\sigma R^2 ln\left[(l/a) + \sqrt{(l/a)^2 - 1}\right]} < \omega < \frac{2}{a^2\mu\sigma}$$

For this range to exist, the conductor spacing must be large enough compared to their radii that

$$1 \ll ln\left[\left(\frac{l}{a}\right) + \sqrt{(l/a)^2 - 1}\right]$$

Because of the logarithmic dependence, the quantity on the right is not likely to be very large.

**14.9.2** From (14.9.11),

$$R=\frac{1}{\sigma 2\pi a\Delta}+\frac{1}{\sigma\pi b^2}=\frac{1}{\pi\sigma}\left[\frac{1}{a\Delta}+\frac{1}{b^2}\right]$$

while, from Prob. 14.2.1

$$L=\frac{\mu_o}{2\pi}ln(a/b);\qquad C=\frac{2\pi\epsilon}{ln(a/b)}$$

For the skin depth to be large compared to the transverse dimensions of the conductors

$$\delta\equiv\sqrt{\frac{2}{\omega\mu\sigma}}\gg\Delta\quad\text{or}\quad b\Rightarrow\omega\ll 2/b^2\mu\sigma\quad\text{and}\quad 2/\Delta^2\mu\sigma$$

This puts an upper limit on the frequency for which the model is valid. To be useful, the model should be valid at sufficiently high frequencies that the inductive reactance can dominate the resistance. Thus, it should extend to

$$\frac{\omega L}{R}=\frac{\omega\mu_o\sigma}{2}ln(a/b)/\left[\frac{1}{a\Delta}+\frac{1}{b^2}\right]>1$$

For the frequency range to include this value but not exceed the skin depth limit,

$$\frac{2\left[\frac{1}{a\Delta}+\frac{1}{b^2}\right]}{\mu\sigma ln(a/b)}\ll\omega\ll\frac{2}{b^2\mu\sigma}\quad\text{and}\quad\frac{2}{\Delta^2\mu\sigma}$$

which is possible only if

$$1\ll\frac{ln(a/b)}{\left[\frac{1}{a\Delta}+\frac{1}{b^2}\right](b^2\text{and}\Delta^2)}$$

Because of the logarithms dependence of $L$, this is not a very large range.

**14.9.3** Comparison of (14.9.18) and (10.6.1) shows the mathematical analogy between the charge diffusion line and one-dimensional magnetic diffusion. The analogous electric and magnetic variables and parameters are

$$H_z\leftrightarrow V,\quad K_p\leftrightarrow V_p,\quad \mu\sigma\leftrightarrow RC,\quad b\leftrightarrow l,\quad x\leftrightarrow z$$

Because the boundary condition on $V$ at $z=0$ is the same as that on $H_z$ at $x=0$, the solution is found by following the steps of Example 10.6.1. From 10.6.21, it follows that the desired distribution of $V$ is

$$V=-V_p\frac{z}{l}-\sum_{n=1}^{\infty}2V_p\frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi z}{l}\right)e^{-t/\tau_n};\quad \tau_n\equiv\frac{RCl^2}{(n\pi)^2}$$

This transient response is represented by Fig. 10.6.3a where $H_z/K_p\rightarrow V/V_p$ and $x/b\rightarrow z/l$.

**14.9.4** See solution to Prob. 10.6.2 using analogy described in solution to Prob. 14.9.3.